

Knot Physics – an Ultimate Unified Theory of Matter and its Motion

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A Theory of Everything (TOE) is physics theory that unifies all the fundamental interactions of nature: gravitation, strong interaction, weak interaction, and electromagnetism. Now, TOE becomes the Holy Grail of modern physics. In this paper, knot physics becomes a new candidate of TOE that not only unified all the fundamental interactions but also explores the underline physics of quantum mechanics. In knot physics, our universe is a standard knot-crystal, a particular periodic entanglement-pattern between two 3-branes (three dimensional branes/manifolds), of which the low energy effective theory not only reproduces the Standard model – an $SU_{\text{Strong}}(3) \otimes SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge theory but also leads to the physics of general relativity. The collective motions of the standard knot-crystal are described by fermionic elementary particles and gauge fields. Fermionic elementary particles are topological excitations that correspond to different types of knots.

I. INTRODUCTION

Physics is knowledge of nature that involves the study of matter and its motion. To understand the universe, people try to answer the following six questions: 1) *what is matter?* 2) *How matter moves?* 3) *what is quantum field?* 4) *what is (local) gauge symmetry?* 5) *what is gravity?* 6) *what is curved spacetime?*

According to Newton's theory, matter is point-like object with mass that obeys classical mechanics, between which there exists the inverse-square law of universal gravitation interaction. Field is another important concept in physics. By understanding electromagnetism, Maxwell introduced the concept of the electromagnetic field and believed that the propagation of light required a medium for the waves, named the luminiferous aether. Thomson came to the idea that classical atoms were knots of swirling vortices in the luminiferous aether. Chemical elements correspond to knots and links. In 20th century, revolution in physics occurred. The concepts in classical physics are changed. Quantum physics including quantum mechanics and quantum field theory describe the universe. In the framework of quantum field theory, it is believed that the interaction comes from exchanging virtual bosons via local gauge symmetry principles. The *Standard model* (an $SU_{\text{Strong}}(3) \otimes SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge theory with Higgs mechanism due to spontaneous symmetry breaking) is a special type of quantum field theory that focuses on three non-gravitational interactions: weak interaction, strong interaction, and electromagnetic interaction[1]. In quantum physics, the matter becomes elementary particles in the Standard Model. On the other hand, to understand gravitational interaction, the general relativity is developed by Einstein that provides a unified description of gravity as a geometric property of spacetime.

Now, Theory of Everything (TOE) to unify all interactions is a dream of physicists. There are several approaches towards TOE. String theory is a possible theory of TOE[2]. According to string theory, quantum matter consists of vibrating strings (or strands) and different oscillatory patterns of strings become different particles with different masses. In condensed matter physics, the idea of our universe as an “*emergent*” phenomenon has become increasingly popular. In emergence approach, a deeper and unified understanding of the universe is developed based on a complicated many-body system. Different quantum fields correspond to different many-body systems: the vacuum corresponds to the ground state and the elementary particles correspond to the excitations of the systems. According to string-net picture proposed by Wen, our universe (such as gauge interaction, Fermi statistics, ...) emerges from a frustrated quantum spin model[3]. In addition, there exist many other proposals of TOE from different points of view, such as loop quantum gravity theory[4], G. Lisi's E8 theory[5], C. Schiller's Strand Model[6], ...

To reach the TOE, we will answer above six questions within a unified picture and point out a new approach towards understanding our universe. We call the new theory *knot physics*. In knot physics, our universe is a particular knot-crystal, a periodic entanglement-pattern between two 3-branes (three dimensional branes/manifolds). The collective motions of knot-crystal are described by fermionic elementary particles and gauge fields. Fermionic elementary particles are topological excitations that correspond to different types of knots. In particular, all four types of fundamental interactions (electromagnetic, weak, strong and gravitational interactions) are unified into single simple framework – the standard knot-crystal that becomes the “holy grail” in physics – the final theory of fundamental physics. Therefore, knot physics fundamentally changes our understanding of the universe.

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The paper is organized as below. In Sec. II, we focus on question 1 and 2 and explore the underline physics of quantum mechanics. In this section, matter and its motion are unified into a simple phenomenon – entanglements between two branes/manifolds. It is known that statistical physics uncovers the underline physics of thermodynamics. Now, knot mechanics uncovers the underline physics of quantum mechanics. In Sec. III, we focus on question 3 and 4 and a new approach towards TOE is proposed that is called – knot-crystal theory. From knot-crystal theory, the knot-crystal becomes fundamental physical object, of which elementary excitations are gauge fields and fermionic particles (knots). In particular, for a special knot-crystal – the standard knot-crystal, the low energy effective theory is just the Standard model – an $SU_{\text{Strong}}(3) \otimes SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge theory with Higgs mechanism due to spontaneous symmetry breaking. In Sec. IV, we focus on question 5 and 6. In this section, we point out that gravity may be regarded as a low energy effective phenomenon in the framework of three dimensional double-helix knot-crystal with very-slowly varying chiral-density-wave order (we call it chiral-spacetime-crystal). The low energy effective theory of deformed chiral-spacetime-crystal is called chiral-gravity theory that reproduces the physics of general relativity. Finally, the conclusions are drawn in Sec. V.

II. KNOT MECHANICS

Quantum mechanics is a successful theory that agrees very well with experiments and provides an accurate description of the dynamic behavior of microcosmic objects. In quantum mechanics, the energy is quantized and can only change by discrete amounts, i.e. $E = \hbar\omega$ where \hbar is Planck constant. There are several fundamental principles in quantum mechanics: wave-particle duality (objects exhibit both 'wave-like' behaviors and 'particle-like' behavior), uncertainty principle (attempting to measure one attribute such as velocity or position may cause another attribute to become less measurable), and superposition principle (a wave-function superimposes multiple co-existing states that have different probabilities). The Schrödinger equation describes how wave-functions evolve, playing a role similar to Newton's second law in classical mechanics. In quantum mechanics, when considering special relativity, the Schrödinger equation is replaced by the Dirac equation.

However, quantum mechanics is far from being well understood. There are a lot of unsolved mysteries in quantum mechanics including quantum entanglement problem and quantum measurement problem. Einstein said, "*Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the 'old one'. I, at any rate, am convinced that He does not throw dice.*" The exploring of the underline physics of quantum mechanics is going on since its establishment[7]. There are a lot of attempts, such as De Broglie's pivot-wave theory[8], the Bohmian mechanics[9], the many-world theory[10], the Nelsonian Mechanics[11] and the idea of primary state diffusion[12]. However, for all these interpretations of quantum mechanics, people focus on the issue of quantum motion but miss the first problem, what is matter (or reality)?

In this section, we try to answer the first two questions (question 1 and question 2) within a unified picture and develop a new theory towards understanding quantum mechanics. We call the new theory – *knot mechanics*. According to knot mechanics, matter is really entanglement between two three dimensional (3D) branes (3-branes) in five dimensional (5D) space and elementary particles are knots, half-winding of a 3-brane around the other. Knots obey knot mechanics. In knot mechanics, quantum motion of knots comes from brane-twisting and classical mechanics comes from knot-shifting. Under infinite thermalization limit, knot mechanics becomes quantum mechanics, in which matter and its motion are unified into a simple phenomenon – entanglement of two 3-branes.

This part is organized as below. In Sec. A, we give the mathematical definition of knot – elementary entanglement between d -branes and introduce knot-operation and knot-number operator. In Sec. B, the complex-field- d -Brane correspondence is shown that enables us to learn the brane-entanglement under usual classical field theory. And in this section we point out that from the point view of classical complex field, a knot that is π -phase-changing on projected 1-brane corresponds to a topological configuration of classical complex field in d dimensional space. In Sec. C, we show two types of motions of single knot – knot-shifting and brane-twisting. Under infinite thermalization condition, there is no knot-shifting and knots do the motion of brane-twisting. In Sec. D, to characterize the dynamics of knots under infinite thermalization condition, we formulate knot mechanics – a unified theory of quantum mechanics and classical mechanics. By using the knot mechanics, the mysteries of quantum mechanics are solved. In Sec. E, two fundamental principles of knot mechanics are given that are matter-motion complementarity principle and quantum-classical contradictoriness. In the end, we also discuss the measurement theory in knot mechanics. Finally, the conclusions are drawn in Sec. F.

A. Knot – elementary entanglement between d -branes

Elementary particles (for example, electrons) are tiny point-like objects with several physical degrees of freedom, including, spin, charge, momentum, mass. It is natural to wonder – *what are elementary particles and could they be divided into smaller parts?* These questions are key to understand quantum mechanics.

In this paper we point out that the elementary particle in physics (for example electron) is knot – an entanglement between two d -branes in $d + 2$ dimensional ($d + 2$ D) space. In physics, the d -branes can be viewed as vortex-lines in $d + 2$ D superconductors (SC)[13]. Because a d -brane is an extended object, a dynamic brane has infinite degrees of freedom. For this reason, we don't study the dynamics of branes. Instead, we consider the *entanglements* between two branes as physics objects that have finite degrees of freedom.

A d dimensional knot is a (local) half-winding of one d -brane around another in $d+2$ D space. Since a knot is an element entanglement between two branes, we focus on the dynamics of knots rather than the dynamics of d -branes. When looking up a knot far away, people find a tiny ball; When approaching it, people see its detailed structure – the entanglement between two entangled d -branes. According to topological nature of a knot, it could not be divided into smaller parts.

1. d -Brane and brane-projection

In this section we define the concept of knots in mathematic.

A d -brane in $d + 2$ dimensional space $\{x_1, x_2, \dots, x_d, x_{d+1}, x_{d+2}\}$ is defined by the following equation

$$\begin{pmatrix} x_{d+1} \\ x_{d+2} \end{pmatrix} = \begin{pmatrix} \xi(x_1, x_2, \dots, x_d) \\ \eta(x_1, x_2, \dots, x_d) \end{pmatrix}. \quad (1)$$

So we can use the two-component function $\begin{pmatrix} \xi(x_1, x_2, \dots, x_d) \\ \eta(x_1, x_2, \dots, x_d) \end{pmatrix}$ to characterize the d -brane. For two d -branes (d -brane-A and d -brane-B) described by $\begin{pmatrix} \xi_A(x_1, x_2, \dots, x_d) \\ \eta_A(x_1, x_2, \dots, x_d) \end{pmatrix}$ and $\begin{pmatrix} \xi_B(x_1, x_2, \dots, x_d) \\ \eta_B(x_1, x_2, \dots, x_d) \end{pmatrix}$, a knot is a local half-winding of one brane around another along an arbitrary direction in $d + 2$ D space. In mathematic, we define a knot by counting the crossing between two branes along a given direction θ in $\{x_{d+1}, x_{d+2}\}$ space and along a given direction \vec{e} in $\{x_1, x_2, \dots, x_d\}$ space.

We then introduce the concept of *brane-projection*. After brane-projection along a given direction, a d -brane is projected into a 1-brane (that is string with infinite length). To describe a knot rigorously, we define three types of brane-projections: an *on-brane-projection* that denotes a projection of a brane along a given direction \vec{e} on $\{x_1, x_2, \dots, x_d\}$ space; an *out-brane-projection* that denotes a projection of a brane along a given direction θ on $\{x_{d+1}, x_{d+2}\}$ space; a *full-brane-projection* that is combination of an *on-brane-projection* and an *out-brane-projection*. In the following parts we use $\hat{P}_{\vec{e}}$, \hat{P}_{θ} , \hat{P}_F to denote the projection operators of on-brane-projection, out-brane-projection, full-brane-projection, respectively.

Firstly, we define on-brane-projection, that is a projection of a d -brane $\begin{pmatrix} \xi(x_1, x_2, \dots, x_d) \\ \eta(x_1, x_2, \dots, x_d) \end{pmatrix}$ onto 1-brane along a given direction \vec{e} on brane. In mathematic, on-brane-projection is

$$\begin{aligned} \hat{P}_{\vec{e}} \begin{pmatrix} \xi(x_1, x_2, \dots, x_d) \\ \eta(x_1, x_2, \dots, x_d) \end{pmatrix} &= \begin{pmatrix} \xi(x'_1, (x'_2)_0, \dots, (x'_d)_0) \\ \eta(x'_1, (x'_2)_0, \dots, (x'_d)_0) \end{pmatrix} \\ &= \begin{pmatrix} \xi(x'_1) \\ \eta(x'_1) \end{pmatrix} \end{aligned} \quad (2)$$

where $x'_1 = \vec{e} \cdot \vec{x}$ is variable on the axis along a given direction \vec{e} on d -brane and $(x'_2)_0, \dots, (x'_d)_0$ are constant on the axes along other orthotropic directions on d -brane. The relationship between the coordinate $(x'_1, x'_2, x'_3, \dots, x'_d)$ and the coordinate $(x_1, x_2, x_3, \dots, x_d)$ is

$$\begin{pmatrix} x'_1 \\ x'_2 \\ \dots \\ x'_d \end{pmatrix} = \hat{R} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_d \end{pmatrix} \text{ where } \hat{R} \text{ is spacial rotation operator. Here } \begin{pmatrix} \xi(x'_1, (x'_2)_0, \dots, (x'_d)_0) \\ \eta(x'_1, (x'_2)_0, \dots, (x'_d)_0) \end{pmatrix} = \begin{pmatrix} \xi(x'_1) \\ \eta(x'_1) \end{pmatrix} \text{ describes a 1-brane in } (x_1, x_2, x_3, \dots, x_d) \text{ space. F}$$

Secondly, we define out-brane-projection, that is a projection of a d -brane $\begin{pmatrix} \xi(x_1, x_2, \dots, x_d) \\ \eta(x_1, x_2, \dots, x_d) \end{pmatrix}$ onto another d -brane. In mathematic, out-brane-projection is defined by

$$\hat{P}_\theta \begin{pmatrix} \xi(x_1, x_2, \dots, x_d) \\ \eta(x_1, x_2, \dots, x_d) \end{pmatrix} = \begin{pmatrix} \xi_\theta(x_1, x_2, \dots, x_d) \\ [\eta_\theta(x_1, x_2, \dots, x_d)]_0 \end{pmatrix} \quad (3)$$

where $\xi_\theta(x_1, x_2, \dots, x_d)$ is variable and $[\eta_\theta(x_1, x_2, \dots, x_d)]_0$ is constant. Because the projection direction out of d -brane is characterized by an angle θ in $\{x_{d+1}, x_{d+2}\}$ space, we have

$$\begin{pmatrix} \xi_\theta \\ \eta_\theta \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad (4)$$

where θ is angle, i.e. $\theta \bmod 2\pi = 0$. So the out-brane-projected d -brane is described by the function

$$\begin{aligned} \xi_\theta(x_1, x_2, \dots, x_d) &= \xi(x_1, x_2, \dots, x_d) \cos \theta \\ &+ \eta(x_1, x_2, \dots, x_d) \sin \theta. \end{aligned} \quad (5)$$

Thirdly, we define the full-brane-projection, that is the combination of an on-brane-projection and an out-brane-projection. In mathematic, the projection operator of full-brane-operation is defined by

$$\hat{P}_F = \hat{P}_\theta \hat{P}_e. \quad (6)$$

After doing a full-brane-projection, a d -brane corresponds to a projected 1-brane described by

$$\begin{aligned} \hat{P}_F \begin{pmatrix} \xi(x_1, x_2, \dots, x_d) \\ \eta(x_1, x_2, \dots, x_d) \end{pmatrix} &= \begin{pmatrix} \xi_\theta(x'_1, (x'_2)_0, \dots, (x'_d)_0) \\ [\eta_\theta(x'_1, (x'_2)_0, \dots, (x'_d)_0)]_0 \end{pmatrix} \\ &= \begin{pmatrix} \xi_\theta(x'_1) \\ [\eta_\theta(x'_1)]_0 \end{pmatrix} \end{aligned} \quad (7)$$

where $\xi_\theta(x'_1)$ is variable and $[\eta_\theta(x'_1)]_0$ is constant. As a result, the full-brane-projected 1-brane is described by $\xi_\theta(x'_1)$.

2. Knot as half-winding between two d -branes

Based on brane-projection, we define the knots between two d -branes (d -brane-A and d -brane-B), $\begin{pmatrix} \xi_A(x_1, x_2, \dots, x_d) \\ \eta_A(x_1, x_2, \dots, x_d) \end{pmatrix}$ and $\begin{pmatrix} \xi_B(x_1, x_2, \dots, x_d) \\ \eta_B(x_1, x_2, \dots, x_d) \end{pmatrix}$. After doing full-brane-projection,

$$\hat{P}_F \begin{pmatrix} \xi_A(x_1, x_2, \dots, x_d) \\ \eta_A(x_1, x_2, \dots, x_d) \end{pmatrix} = \begin{pmatrix} \xi_{A,\theta}(x'_1) \\ [\eta_{A,\theta}(x'_1)]_0 \end{pmatrix} \quad (8)$$

and

$$\hat{P}_F \begin{pmatrix} \xi_B(x_1, x_2, \dots, x_d) \\ \eta_B(x_1, x_2, \dots, x_d) \end{pmatrix} = \begin{pmatrix} \xi_{B,\theta}(x'_1) \\ [\eta_{B,\theta}(x'_1)]_0 \end{pmatrix}, \quad (9)$$

we have two projected 1-branes that are described by $\xi_{A,\theta}(x'_1)$ and $\xi_{B,\theta}(x'_1)$, respectively.

A knot is half-winding entanglement between two d -branes. Based on the projected 1-branes, we define a knot by a monotonic function $F_\theta(x'_1) = \xi_{A,\theta}(x'_1) - \xi_{B,\theta}(x'_1)$ with

$$\text{sgn} [F_\theta(x'_1 \rightarrow -\infty) \cdot F_\theta(x'_1 \rightarrow \infty)] = -1 \quad (10)$$

where $\text{sgn} [x] = \frac{x}{|x|}$. From the topological character of a knot, there must exist a point, at which $\xi_{A,\theta}(x'_1)$ is equal to $\xi_{B,\theta}(x'_1)$. The position of the point $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_d)$ is determined by a local solution of the *knot-equation*

$$F_\theta(x'_1) = 0 \quad (11)$$

or

$$\xi_{A,\theta}(x'_1) = \xi_{B,\theta}(x'_1). \quad (12)$$

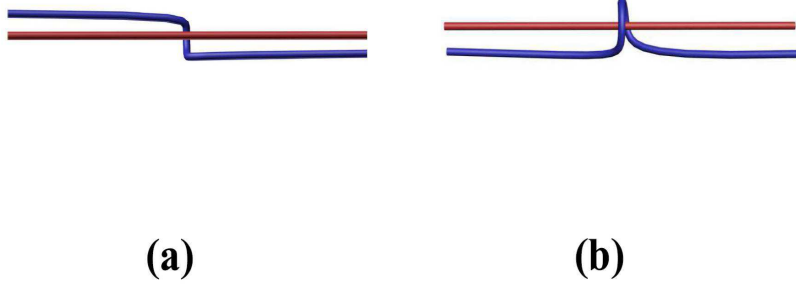


FIG. 1: (a) A knot as half-winding between two 1-branes; (b) A winding between two 1-branes, of which there are two knots.

Each solution $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_d)$ corresponds to a knot-solution. Because the solution $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_d)$ depends on the directions of projection, \vec{e} or θ , we have $(\bar{x}_1(\vec{e}, \theta), \bar{x}_2(\vec{e}, \theta), \dots, \bar{x}_d(\vec{e}, \theta))$. In addition, there are always *two degenerate* knot-solutions at a point $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_d)$: for one solution, we have $\eta_{A,\theta}(x'_1) > \eta_{B,\theta}(x'_1)$; for the other solution, we have $\eta_{A,\theta}(x'_1) < \eta_{B,\theta}(x'_1)$.

To identify a knot (local half-winding between two d -branes), there are three conditions for the knot-solutions:

- 1) *conservation condition*: The conservation condition indicates that a knot is a topological entanglement between two d -branes. The existence of a knot-solution is independent on the directions of projection, \vec{e} or θ . When one gets a knot-solution along a given direction, it will never split or disappear whatever changing the projection direction.
- 2) *isotropic condition*: According to isotropic condition, the knot is rotation-invariant and has the same structure along different directions in d -dimensional space.
- 3) *locality condition*: The locality condition indicates that a knot is a local entanglement between two d -branes. The variety of the position of a knot-solution is finite, i.e.

$$\max \Delta x \leq a \quad (13)$$

where a is finite distance in $(x_1, x_2, x_3, \dots, x_d)$ space and

$$\Delta x = \sqrt{(\bar{x}_1(\vec{e}, \theta) - \bar{x}_1(\vec{e}', \theta'))^2 + (\bar{x}_2(\vec{e}, \theta) - \bar{x}_2(\vec{e}', \theta'))^2 + \dots (\bar{x}_d(\vec{e}, \theta) - \bar{x}_d(\vec{e}', \theta'))^2} \quad (14)$$

is the distance between two arbitrary knot-solutions with different projection angle θ, θ' , $(\bar{x}_1(\vec{e}, \theta), \bar{x}_2(\vec{e}, \theta), \dots, \bar{x}_d(\vec{e}, \theta))$ and $(\bar{x}_1(\vec{e}', \theta'), \bar{x}_2(\vec{e}', \theta'), \dots, \bar{x}_d(\vec{e}', \theta'))$. The value of a characterizes the size of the region, where half-winding between two branes occurs. To emphasize this point of view, we call this region as *winding region*. When one is far away, a knot looks like a point-like object; when one closes, a knot looks like a d -dimensional "fuzzy" ball. To avoiding the "observer-dependent", we define the average position of a knot

$$(\langle \bar{x}_1(\vec{e}, \theta) \rangle, \langle \bar{x}_2(\vec{e}, \theta) \rangle, \dots, \langle \bar{x}_d(\vec{e}, \theta) \rangle). \quad (15)$$

In summary, for a half-winding between two d -branes, there exists a knot-solution satisfying conservation condition, isotropic condition and locality condition. We call such entanglement structure a knot between two d -branes. Fig.1.(a) is an illustration of knot as half-winding between two 1-branes and Fig.2.(b) shows a winding between two 1-branes, of which there are two knots.

3. Complex representation for knots

In above section, we give a generalized definition of a knot for two entangled d -branes. To characterize a generalized knot, we introduce a complex representation of d -brane on ξ - η complex plane,

$$z = x_{d+1} + ix_{d+2} = \eta + i\xi = re^{i\phi} \quad (16)$$

where r is amplitude and ϕ is angle in ξ - η complex plane. In the following part, we call the twisted angle $\phi(x)$ by "phase angle" and $e^{i\phi(x)}$ by "phase". Then, the two d -branes (A and B) are described by

$$\begin{aligned} z_A(x_1, x_2, \dots, x_d) &= \eta_A(x_1, x_2, \dots, x_d) + i\xi_A(x_1, x_2, \dots, x_d) \\ &= r_A(x_1, x_2, \dots, x_d) \exp[i\phi_A(x_1, x_2, \dots, x_d)] \end{aligned} \quad (17)$$

and

$$\begin{aligned} z_B(x_1, x_2, \dots, x_d) &= \eta_B(x_1, x_2, \dots, x_d) + i\xi_B(x_1, x_2, \dots, x_d) \\ &= r_B(x_1, x_2, \dots, x_d) \exp[i\phi_B(x_1, x_2, \dots, x_d)], \end{aligned} \quad (18)$$

respectively. We call $z_i(x_1, x_2, \dots, x_d)$ *knot-functions*. By a complex representation of d -brane, the knot-equation becomes

$$\hat{P}_F[z_A(x_1, x_2, \dots, x_d)] = \hat{P}_F[z_B(x_1, x_2, \dots, x_d)]. \quad (19)$$

We then take entanglements between a flat d -brane and a winding d -brane with fixed distance (r_0) as example.

A flat d -brane is described by

$$\begin{pmatrix} \xi_A(x_1, x_2, \dots, x_d) \\ \eta_A(x_1, x_2, \dots, x_d) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (20)$$

or

$$z_A(x_1, x_2, \dots, x_d) = 0 \quad (21)$$

and a winding brane is described by

$$\begin{pmatrix} \xi_B(x_1, x_2, \dots, x_d) \\ \eta_B(x_1, x_2, \dots, x_d) \end{pmatrix} = r_0 \begin{pmatrix} \cos \phi_B(x_1, x_2, \dots, x_d) \\ \sin \phi_B(x_1, x_2, \dots, x_d) \end{pmatrix} \quad (22)$$

or

$$z_B(x_1, x_2, \dots, x_d) = r_0 \exp[i\phi_B(x_1, x_2, \dots, x_d)]. \quad (23)$$

Because the knot comes from winding of d -brane-B, we focus on the deformation of d -brane-B. A knot between the two d -branes along a given direction \vec{e} is defined by

$$\phi_B(x'_1 \rightarrow \infty) - \phi_B(x'_1 \rightarrow -\infty) = \pm\pi \quad (24)$$

where x'_1 is the coordination on the axis along a given direction \vec{e} . As a result, the knot is a π -domain wall on the axis along a given direction \vec{e} . In the following parts, we define a knot with $\phi_B(x'_1 \rightarrow \infty) - \phi_B(x'_1 \rightarrow -\infty) = \pi$ to a knot of clockwise winding and a knot with $\phi_B(x'_1 \rightarrow \infty) - \phi_B(x'_1 \rightarrow -\infty) = -\pi$ to a knot of anti-clockwise winding.

Because of

$$\hat{P}_F[z_A(x_1, x_2, \dots, x_d) = 0] = \xi_{A,\theta}(x'_1) = 0, \quad (25)$$

the projected flat d -brane is a straight 1D line along \vec{e} direction, we could consider it to be the axis along this direction. On the other hand,

$$\begin{aligned} \hat{P}_F[z_B(x_1, x_2, \dots, x_d)] &= \xi_{B,\theta}(x'_1) \\ &= r_0 \cos[\phi_B(x'_1) - \theta] \end{aligned} \quad (26)$$

is curved 1D line. Here θ is the angle for out-brane-projection, From the topological property of a knot, there must exist a point, at which $\xi_{B,\theta}(x'_1) = r_0 \cos[\phi_B(x'_1) - \theta]$ is equal to $\xi_{A,\theta}(x'_1) = 0$. The knot-equation becomes

$$\xi_{A,\theta}(x'_1) = \xi_{B,\theta}(x'_1) \quad (27)$$

or

$$\cos[\phi_B(x'_1) - \theta] = 0. \quad (28)$$

As a result, we get the knot-solution

$$\phi_B(x'_1) = \pm \frac{\pi}{2} + \theta \quad (29)$$

where x'_1 is variable along \vec{e} direction and θ is the constant that depends on the projection direction in $\{x_{d+1}, x_{d+2}\}$ space.

4. Linear knot between a flat d -brane and a winding d -brane

In this section, we focus on a special type of knots – *linear knots*[14], that are special entanglements between a flat d -brane (d -brane-A) and a winding d -brane (d -brane-B), of which the knot-functions are given by

$$\begin{aligned} z_A(x_1, x_2, \dots, x_d) &= 0, \\ z_B(x_1, x_2, \dots, x_d) &= r_0 \exp[i\phi_B(x_1, x_2, \dots, x_d)]. \end{aligned} \quad (30)$$

In mathematic, a linear knot along a given direction \vec{e} is defined by

$$\phi_B(x'_1) = \begin{cases} \phi_0 \mp \frac{\pi}{2}, & x'_1 \in (-\infty, x_0] \\ \phi_0 \mp \frac{\pi}{2} \pm k_0(x'_1 - x_0), & x'_1 \in (x_0, x_0 + a] \\ \phi_0 \pm \frac{\pi}{2}, & x'_1 \in (x_0 + a, \infty) \end{cases} \quad (31)$$

where x'_1 is the coordination on the axis along a given direction \vec{e} and ϕ_0 is a constant angle (in general, we may fix the global phase angle ϕ_0 to be zero as $\phi_0 \equiv 0$), $k_0 = \frac{\pi}{a}$. + denotes clockwise winding and – denotes anti-clockwise winding. Here, the word "linear" means a linear relationship between $\phi_B(x'_1)$ and x'_1 as $\phi_B(x'_1) \propto x'_1 - x_0$ in the winding region of $x_0 < x'_1 \leq x_0 + a$ that denotes a half-winding between two d -branes. In next part, we will show the mechanism of linear knot. And for simplicity, we call "linear knot" to be "knot". In the limit of $x'_1 \rightarrow -\infty$, we have $\phi_B(x'_1) = \phi_0 \mp \frac{\pi}{2}$; in the limit of $x'_1 \rightarrow \infty$, we have $\phi_B(x'_1) = \phi_0 \pm \frac{\pi}{2}$. So, we obtain

$$\phi_B(x'_1 \rightarrow \infty) - \phi_B(x'_1 \rightarrow -\infty) = \pm\pi. \quad (32)$$

For a given projection direction \vec{e} and a fixed θ , we get a knot-equation

$$\begin{aligned} \phi_B(x'_1) &= \phi_0 \mp \frac{\pi}{2} \pm k_0(x'_1 - x_0) \\ &= \pm \frac{\pi}{2} + \theta. \end{aligned} \quad (33)$$

The knot-solution is

$$\bar{x}'_1 - x_0 = (\pm \frac{\pi}{2} \pm \frac{\pi}{2} + \theta - \phi_0)/k_0. \quad (34)$$

It is obvious that there always exist four knot-solutions: for a clockwise winding knot, we have a knot-solution $\bar{x}'_1 = x_0 + (\theta - \phi_0)/k_0$ or $\bar{x}'_1 = x_0 + (\pi + \theta - \phi_0)/k_0$; for an anti-clockwise winding knot, we have a knot-solution $\bar{x}'_1 = x_0 + (\theta - \phi_0)/k_0$ or $\bar{x}'_1 = x_0 + (-\pi + \theta - \phi_0)/k_0$. The size of winding region is a . The average position of the knot is $\langle \bar{x}'_1(\theta) \rangle = x_0 + \frac{a}{2}$. When looking the linear knot far away, a tiny d -dimensional ball is found, i.e. $\Delta \bar{x}'_1 \gg a$; When approaching it, it becomes half-winding between two d -branes.

Fig.2.(a) is an illustration of a 1D linear knot. In Fig.2.(b) shows the corresponding complex field of a 1D linear knot (the red arrows denote the directions of phase angles). In Fig.2.(c), the relation between phase and coordination of a 1D linear knot is shown.

5. Knot-operation and knot-number operator

To characterize knots, we introduce its operator-representation on projected 1-brane.

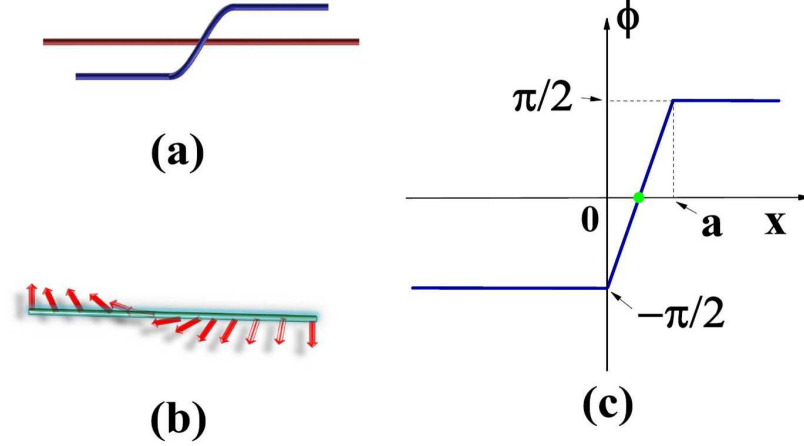


FIG. 2: (a) The illustration of a linear knot; (b) Corresponding complex field of a linear knot (the red arrows denote the directions of phase angles); (c) The relation between phase and coordination of a linear knot. The green spot denotes the average position of the knot.

Firstly, we consider two parallel flat d -branes (d -brane-A and d -brane-B) without knots, that are denoted by

$$\begin{aligned} z_A &= \eta + i\xi = 0, \\ z_B &= \eta + i\xi = r_0 e^{i\phi_0} \end{aligned} \quad (35)$$

where ϕ_0 denotes a constant angle in ξ - η complex plane that can be set to be zero at beginning.

We then introduce knot-operation $\hat{U}(\phi_B(x'_1))$ on d -brane-B and generate a knot along the direction \vec{e} by doing a knot-operation

$$\hat{U}(\phi_B(x'_1)) = \exp[i\phi_B(x'_1) \cdot \hat{K}] \quad (36)$$

where $\hat{K} = -i \frac{d}{d\phi_B}$ is knot-number operator and x'_1 is variable along \vec{e} direction. The knot-number can be obtained by the following equation

$$\begin{aligned} \langle \hat{K} \rangle &= \frac{1}{\pi(z_B)^2} \int z_B^*|_{x'_1=x_0} (\phi_B(x'_1)) \\ &\hat{K} z_B|_{x'_1=x_0} (\phi_B(x'_1)) d\phi_B \end{aligned} \quad (37)$$

where $z_B^*|_{x'_1=x_0} (\phi_B(x'_1))$ is complex conjugation of $z_B|_{x'_1=x_0} (\phi_B(x'_1))$. In physics, $\langle \hat{K} \rangle$ measures the total phase changing for linear knot along the direction \vec{e} .

For a linear knot, we have

$$\phi_B(x'_1) = \begin{cases} -\frac{\pi}{2}, & x'_1 \in (-\infty, x_0] \\ -\frac{\pi}{2} + k_0(x'_1 - x_0), & x'_1 \in (x_0, x_0 + a] \\ \frac{\pi}{2}, & x'_1 \in (x_0 + a, \infty) \end{cases} \quad (38)$$

where $k_0 = \frac{\pi}{a}$. We use $z_B|_{x'_1=x_0} (\phi_B(x'_1)) = \hat{U}(\phi_B(x'_1))z_B$ to denote d -brane-B winding a flat one (d -brane-A) around the point $x'_1 = x_0 + \frac{a}{2}$. For a linear knot, the knot-number $\langle \hat{K} \rangle$ is obtained as

$$\begin{aligned} \langle \hat{K} \rangle &= \frac{-i}{\pi} \int \hat{U}^*(\phi_B(x)) \frac{d\hat{U}(\phi_B(x))}{d\phi_B} d\phi_B \\ &= \frac{1}{\pi} \int_0^\pi d\phi_B = 1. \end{aligned}$$

B. Complex-field- d -Brane correspondence and topological nature of knots

1. Complex-field- d -Brane correspondence

In this section, we point out that the brane-functions $z_i(x_1, x_2, \dots, x_d) = r_i(x_1, x_2, \dots, x_d) \exp(i\phi_i(x_1, x_2, \dots, x_d))$ for a winding brane can be mapped onto a classical complex field $\Phi_i(x_1, x_2, \dots, x_d)$ in d dimensional space as

$$\begin{aligned} z_i(x_1, x_2, \dots, x_d) &= r_i(x_1, x_2, \dots, x_d) \\ &\exp(i\phi_i(x_1, x_2, \dots, x_d)) \\ &\implies \Phi_i(x_1, x_2, \dots, x_d). \end{aligned} \quad (39)$$

For example, a flat d -brane corresponds to a constant complex field that is denoted by $\Phi_i(x_1, x_2, \dots, x_d) = 0$ and a winding d -brane nearby corresponds to a "condensed" complex field with constant amplitude, i.e. $|\Phi_i(x_1, x_2, \dots, x_d)| = r_0$.

The *complex-field- d -Brane correspondence* enables us to learn the brane-entanglement under a familiar framework – the usual classical field theory. In particular, from the point view of classical complex field, *a knot that is π -phase-changing on projected 1-brane corresponds to a topological configuration of classical complex field in d dimensional space*: 1D knot is a π -domain wall; 2D knot is a 2π -flux; 3D knot is a Dirac monopole.

In this section we will show the topological properties of single knot of two entangled d -branes in $d + 2$ D space ($d = 1, 2, 3$).

2. 1D linear knot

Firstly we discuss the case of single linear knot between two 1-branes in 3D space (1-brane-A and 1-brane-B). In Fig.2, we show a knot that is half-winding between two 1-branes. The two 1-branes are denoted by $z_A(x) \rightarrow \Phi_A(x) = 0$ and $z_B(x) \rightarrow \Phi_B(x) = r_0 e^{i\phi_0 + i\phi(x)}$.

For a linear knot at given point, there are two degenerate classical complex fields $\Phi_{a,B}(x) = r_0 e^{i\phi_0 + i\phi_a(x)}$ ($a = 1, 2$) that correspond to

$$\phi_B(x) = \begin{cases} -\frac{\pi}{2}, & x \in (-\infty, x_0] \\ -\frac{\pi}{2} + k_0(x - x_0), & x \in (x_0, x_0 + a] \\ \frac{\pi}{2}, & x \in (x_0 + a, \infty) \end{cases} \quad (40)$$

and

$$\phi_B(x) = \begin{cases} \frac{\pi}{2}, & x \in (-\infty, x_0] \\ \frac{\pi}{2} - k_0(x - x_0), & x \in (x_0, x_0 + a] \\ -\frac{\pi}{2}, & x \in (x_0 + a, \infty) \end{cases} \quad (41)$$

where ϕ_0 is a constant phase that is set to be zero and $k_0 = \frac{\pi}{a}$. On 1-brane, we have $\hat{P}_F[z_i(x)] = \hat{P}_\theta[z_i(x)]$. According to $\hat{P}_\theta[z_A(x)] = \hat{P}_\theta[z_B(x)]$, we derive $k_0(\bar{x} - x_0) - \theta = 0$ or $k_0(\bar{x} - x_0) - \theta = \pi$ where \bar{x} is a point of the knot-solution. As a result, in the winding region ($x_0 < x \leq x_0 + a$) the knot is at $\bar{x} = x_0 + \theta/k_0$ or $\bar{x} = x_0 + (\pi + \theta)/k_0$. The complex field $\Phi_B(x)$ sharply changes in this region; Out of the winding region, the knot-function is constant, $z_B(x) = \pm i \cdot r_0$ and complex field $\Phi_{a,B}(x)$ is a constant.

A 1D linear knot can be regarded as domain wall of π -phase shifting in a 1D condensed complex field. We may define a topological number $\mathcal{N}_{1D-Knot}$ for a 1D linear knot

$$\begin{aligned} \mathcal{N}_{1D-Knot} &= \frac{1}{(r_0)^2 \pi} \int_{-\infty}^{\infty} \Phi_{a,B}^*(x) d\Phi_{a,B}(x) \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\phi(x)}{dx} dx \\ &= \frac{1}{\pi} \int d\phi(x) = \pm 1. \end{aligned} \quad (42)$$

The topological number $\mathcal{N}_{1D-Knot}$ that characterizes a π -domain wall of classical complex field $\Phi_B(x)$ is obviously independent on the global phase angle ϕ_0 . See the illustration in Fig.2.

3. 2D linear knot

Next, we consider 2D linear knot between two 2-branes (2-brane-A and 2-brane-B) in 4D $\{x, y, \xi, \eta\}$ space, that are described by two classical complex fields $z_A(x, y) \rightarrow \Phi_A(x, y) = 0$ and $z_B(x, y) \rightarrow \Phi_{a,B}(x, y) = r_0 e^{i\phi_0 + i\phi_a(x, y)}$ ($a = 1, 2$). We use complex plane to describe the 2D space, i.e. $(x, y) = (r, \varphi)$ where r is a distance to the center and φ is spacial angle.

For two 2-branes in 4D space, there exist two types of knots at a given point $(0, 0)$: one denotes clockwise winding, of which the corresponding complex field $\Phi_{1,B}(r, \varphi)$ is

$$\ln\left[\frac{\Phi_{1,B}(r, \varphi)}{r_0}\right] + \phi_0 = \begin{pmatrix} \frac{r}{a}\varphi, & r < a, \\ \varphi, & r \geq a \end{pmatrix}, \quad (43)$$

the other $\Phi_{2,B}(r, \varphi)$ anti-clockwise winding, of which the knot-function is

$$-\ln\left[\frac{\Phi_{2,B}(r, \varphi)}{r_0}\right] + \phi_0 = \begin{pmatrix} \frac{r}{a}\varphi, & r < a \\ \varphi, & r \geq a \end{pmatrix}. \quad (44)$$

ϕ_0 is the global phase of knot that can be set to be zero.

The 2D knot, a local half-winding of two 2-branes in 4D space is π -phase change along a given direction on 2D space. Under on-brane-projection along the direction \vec{e}_y , we have a 1D reduction formula of the 2D knot

$$\ln\left[\frac{\Phi_{1,B}(y)}{r_0}\right] = \begin{cases} -\frac{\pi}{2}, & y \in (-\infty, 0] \\ -\frac{\pi}{2} + \frac{y}{a}\pi, & y \in (0, a] \\ \frac{\pi}{2}, & y \in (a, \infty) \end{cases}, \quad (45)$$

and

$$-\ln\left[\frac{\Phi_{2,B}(y)}{r_0}\right] = \begin{cases} -\frac{\pi}{2}, & y \in (-\infty, 0] \\ -\frac{\pi}{2} + \frac{y}{a}\pi, & y \in (0, a] \\ \frac{\pi}{2}, & y \in (a, \infty) \end{cases}. \quad (46)$$

In the winding region of 2D knot $r < a$, there exist solutions $\pm \frac{\bar{r}}{a} \bar{\varphi} = -\theta$ or $\pm \frac{\bar{r}}{a} \bar{\varphi} = -\theta + \pi$; out of the core of knot, there exists a singular line of knot-function at $\pm \bar{\varphi} = -\theta$ or $\pm \bar{\varphi} = -\theta + \pi$. The knot's position $(\langle \bar{x}(\theta) \rangle, \langle \bar{y}(\theta) \rangle)$ is given by

$$\begin{aligned} \langle \bar{x}(\theta) \rangle &= \frac{1}{2\pi} \int \bar{x}(\theta) d\theta = 0, \\ \langle \bar{y}(\theta) \rangle &= \frac{1}{2\pi} \int \bar{y}(\theta) d\theta = 0. \end{aligned} \quad (47)$$

The 2D knot is a 2π -flux in 2D space. We may define a winding number of a 2D knot to characterize its topological property

$$\begin{aligned} \mathcal{N}_{2D-knot} &= \frac{1}{2\pi} \oint_{\mathcal{C}} \vec{A}(x, y) \cdot d\vec{x} = \frac{1}{2\pi} \oint_{\mathcal{C}} d\phi(x, y) \\ &= \frac{1}{2\pi} \oint_{\mathcal{C}} d\varphi(x, y) = \pm 1 \end{aligned} \quad (48)$$

where \mathcal{C} is a closed loop around knot out of the core and $\vec{A}(x, y) = \vec{\nabla} \phi(x, y)$. That means when we generate a 2D linear knot with a fixed phase ϕ_0 . The winding number $\mathcal{N}_{2D-Knot}$ that characterizes a 2π -flux is independent on the global phase angle ϕ_0 .

The 2D linear knot has spacial rotation symmetry on 2D space around its center position $(\langle \bar{x}(\theta) \rangle, \langle \bar{y}(\theta) \rangle) = (0, 0)$. We can rotate the knot-function 90° , $\varphi \rightarrow \varphi + \frac{\pi}{2}$ i.e. x-axis to y-axis and get

$$\begin{pmatrix} \ln\left[\frac{z(r, \varphi)}{r_0}\right] \\ \ln\left[\frac{\bar{z}(r, \varphi)}{r_0}\right] \end{pmatrix} \rightarrow \begin{pmatrix} \ln\left[\frac{z(r, \varphi)}{r_0}\right] - \frac{\pi}{2} \\ \ln\left[\frac{\bar{z}(r, \varphi)}{r_0}\right] + \frac{\pi}{2} \end{pmatrix} \quad (49)$$

or

$$\begin{pmatrix} z(r, \varphi) \\ \tilde{z}(r, \varphi) \end{pmatrix} \rightarrow \begin{pmatrix} z(r, \varphi) \cdot e^{-i\frac{\pi}{2}} \\ \tilde{z}(r, \varphi) \cdot e^{i\frac{\pi}{2}} \end{pmatrix}. \quad (50)$$

We may use spinor-representation to characterize the rotation symmetry of 2D knot-functions, $\mathbf{z} = \frac{1}{r_0} \begin{pmatrix} \Phi_{1,B}(r, \varphi) \\ \Phi_{2,B}(r, \varphi) \end{pmatrix}$ where $\Phi_{1,B}(r, \varphi)$ is knot-function of clockwise winding knot and $\Phi_{2,B}(r, \varphi)$ is knot-function of anti-clockwise winding. We then introduce $O(3)$ rotor-representation by spinor,

$$n^a(x)\sigma^a = \bar{\mathbf{z}}\sigma\mathbf{z} \quad (51)$$

where $[\sigma^a, \sigma^b] = i\epsilon^{abc}\sigma^c$ is an $SU(2)$ “isospin” density operator ($a = 1, 2, 3$). $\mathbf{n} = (n^1(x, y), n^2(x, y), n^3(x, y))$ satisfies $\mathbf{n}^2 = (n^1)^2 + (n^2)^2 + (n^3)^2 = 1$. With the help of $O(3)$ rotor-representation, we get a 2D knot to be a 2D “half-skyrmion”, that is described by

$$\mathbf{n} = \begin{pmatrix} (\frac{\vec{r}}{|\vec{r}|})^{\frac{|\vec{r}|}{a}}, & |\vec{r}| < a \\ \frac{\vec{r}}{|\vec{r}|}, & |\vec{r}| \geq a \end{pmatrix}. \quad (52)$$

4. 3D linear knot

Thirdly, we study 3D linear knot between two 3-branes in 5D $\{x, y, z, \xi, \eta\}$ space (3-brane-A and 3-brane-B), that are described by two classical complex fields $z_A(\vec{r}) \rightarrow \Phi_A(\vec{r}) = 0$ and $z_B(\vec{r}) \rightarrow \Phi_B(\vec{r})$ where \vec{r} denotes the position in 3D space.

For two 3-branes in 5D, there also exist two types of knots at a given point (we choose $(0, 0, 0)$) that are described by two-component (spinor) classical complex fields, $\Phi_{1,B}$ and $\Phi_{2,B}$. $\Phi_{1,B}$ is knot-function of clockwise winding knot and $\Phi_{2,B}$ is knot-function of anti-clockwise winding. We use $\mathbf{z} = \frac{1}{r_0} \begin{pmatrix} \Phi_{1,B}(\vec{r}) \\ \Phi_{2,B}(\vec{r}) \end{pmatrix}$ to characterize the complex phase of a 3D linear knot. The knot-solution under spinor-representation is described by

$$\exp[i \sum_{\vec{r}} \mathbf{z}^* \sigma^a \mathbf{z} \cdot e^a(\vec{r}; \vec{e}) \varphi(\vec{r}; \vec{e})], \quad (53)$$

where $\varphi(\vec{r}; \vec{e}) = \arg(\vec{r}; \vec{e})$ and $e^a(\vec{r}; \vec{e})$ being a unit vector orthogonal to \vec{r} and \vec{e} .

We can also define $O(3)$ rotor by spinor, $n^a(x)\sigma^a = \bar{\mathbf{z}}\sigma\mathbf{z}$ ($a = 1, 2, 3$). In general, out of winding region, for 3D knots at $(0, 0, 0)$, we have the $O(3)$ rotor of the winding 3-brane as

$$\mathbf{n} = \frac{\vec{r}}{|\vec{r}|}, \quad r = |\vec{r}| \geq a. \quad (54)$$

Inside the winding region, the $O(3)$ rotor becomes

$$\mathbf{n} = (\frac{\vec{r}}{|\vec{r}|})^{\frac{|\vec{r}|}{a}}, \quad r = |\vec{r}| < a. \quad (55)$$

The 3D linear knot has rotation symmetry on 3D space. According the rotation symmetry, the knot's position $(\langle \bar{x}(\theta) \rangle, \langle \bar{y}(\theta) \rangle, \langle \bar{z}(\theta) \rangle)$ is given by $(0, 0, 0)$.

Under on-brane-projection along the direction \vec{e}_z , we have

$$\ln[\frac{\Phi_{1,B}(\vec{r})}{r_0}] + \phi_0 = \begin{cases} 0, & z \in (-\infty, 0] \\ k_0 z, & z \in (0, a] \\ \pi, & z \in (a, \infty) \end{cases}, \quad (56)$$

and

$$-\ln[\frac{\Phi_{2,B}(\vec{r})}{r_0}] + \phi_0 = \begin{cases} 0, & z \in (-\infty, 0] \\ k_0 z, & z \in (0, a] \\ \pi, & z \in (a, \infty) \end{cases} \quad (57)$$

where $k_0 = \frac{\pi}{a}$ and ϕ_0 is the global phase of knot.

A 3D knot is really a "magnetic" monopole[15] that shows a hedgehog behavior at $(0,0,0)$. To characterize it topological properties, We may define a topological number for a 3D knot

$$\mathcal{N}_{3D-\text{knot}} = \frac{1}{2\pi} \iint_S \vec{B}(\vec{r}) \cdot d\vec{S} = \pm 1 \quad (58)$$

where $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{a}(\vec{r})$, $a_\mu(\vec{r}) = i\vec{z}(\vec{r})\partial_\mu \mathbf{z}(\vec{r})$, and S is a closed surface surrounding the knot.

In summary, linear knot corresponds to *singularity* of complex classical fields in ξ - η complex plane. Thus, there exists *topological structure* around the knots. In mathematic, the existence of a knot with nontrivial topological number is protected by the homotopy groups as

$$\begin{aligned} \pi_0(\mathbf{Z}_2) &= \mathcal{N}_{1D-\text{Knot}} \neq 0, \quad d=1, \\ \pi_1(\text{U}(1)) &= \mathcal{N}_{2D-\text{Knot}} \neq 0, \quad d=2, \\ \pi_2(\text{SU}(2)/\text{U}(1)) &= \mathcal{N}_{3D-\text{Knot}} \neq 0, \quad d=3 \end{aligned} \quad (59)$$

respectively. In 1D, the boundary $\{-\infty, +\infty\}$ is mapped onto \mathbf{Z}_2 . A 1D linear knot becomes a π -phase *domain wall*; In 2D, the circle at infinity, S_∞^1 , is mapped onto $\text{U}(1)$. A 2D linear knot becomes a 2π -*flux*; In 3D, the sphere at infinity S_∞^2 is mapped onto $\text{SU}(2)/\text{U}(1)$. A 3D linear knot becomes a *Dirac monopole*. For higher dimensions, due to $\pi_{d-1}(\text{SO}(d)/\text{SO}(d-1)) \neq 0$, we have d -dimensional knots.

C. Knot motion – knot-shifting and brane-twisting

There are two types of motions of single knot: *knot-shifting* and *brane-twisting*. Correspondingly, we define m_{shift} , m_{twist} to be shifting-mass and twisting-mass of a knot, respectively.

1. Knot-shifting

Firstly, we discuss the motion from knot-shifting.

According to classical-complex-field- d -Brane correspondence, $z_i(x_1, x_2, \dots, x_d) \implies \Phi_i(x_1, x_2, \dots, x_d)$, we may regard the knots as topological solitons of classical complex field in d -dimensional space, of which the solution is

$$\mathbf{z}(\vec{x} - \vec{x}_0) = \frac{1}{r_0} \begin{pmatrix} \Phi_{1,B}(\vec{x} - \vec{x}_0) \\ \Phi_{2,B}(\vec{x} - \vec{x}_0) \end{pmatrix} \quad (60)$$

where \vec{x}_0 is the position of the knot.

Knot-shifting is a classical motion in d -dimensional space, i.e.

$$\begin{aligned} \mathbf{z}(\vec{x} - \vec{x}_0(t)) \\ = \frac{1}{r_0} \begin{pmatrix} \Phi_{1,B}(\vec{x} - \vec{x}_0(t)) \\ \Phi_{2,B}(\vec{x} - \vec{x}_0(t)) \end{pmatrix} \end{aligned} \quad (61)$$

where the position of the knot becomes time-dependent i.e.,

$$\vec{x}_0(t) = \vec{x}_0 + \vec{v}_{\text{shift}} \cdot t. \quad (62)$$

\vec{v}_{shift} is velocity of knot-shifting and t denotes time. In particular, for knot-shifting the formula of knot-function are fixed. If the mass of knot is m_{shift} , the momentum and the energy of the shifting knot are $\vec{p} = m_{\text{shift}} \cdot \vec{v}_{\text{shift}}$ and $\frac{1}{2}m_{\text{shift}}(\vec{v}_{\text{shift}})^2$, respectively. We can also define the Newton second law, $\vec{F} = m_{\text{shift}} \frac{d\vec{v}_{\text{shift}}}{dt}$ where \vec{F} is the force in d -dimensional space. As a result, knot-shifting is a classical motion. See the illustration in Fig.3.(a).

We take 1D knot as an example to show the knot-shifting. For a shifting 1D knot, we have $\Phi_B(x) = r_0 e^{i\phi_0 + i\phi(x(t))}$ where

$$\phi(x(t)) = \begin{cases} -\frac{\pi}{2}, & x \in (-\infty, x_0(t)] \\ -\frac{\pi}{2} + \frac{\pi}{a}(x - x_0(t)), & x \in (x_0(t), x_0(t) + a] \\ \frac{\pi}{2}, & x \in (x_0(t) + a, \infty) \end{cases} \quad (63)$$

and $x_0(t) = x_0 + v_{\text{shift}} \cdot t$. As shown in Fig.3.(b), the position of knot is determined by the knot-equation

$$\bar{x} = (x_0 + \frac{a}{2}) + v_{\text{shift}} \cdot t. \quad (64)$$

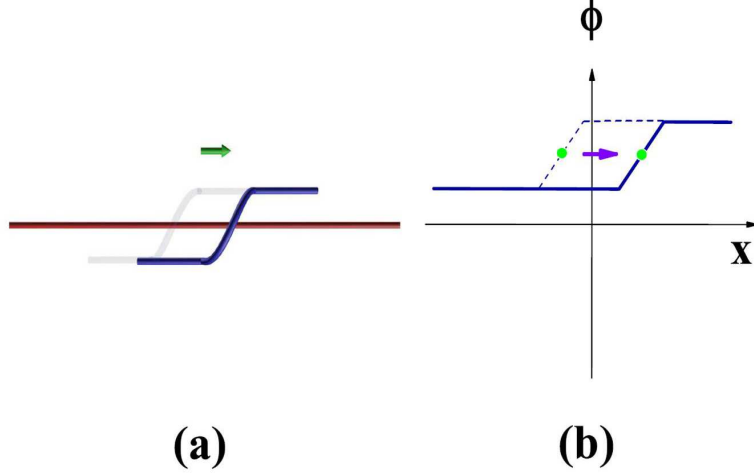


FIG. 3: (a) An illustration for knot-shifting; (b) The relation between phase and coordination for knot-shifting. The green spot denotes the average position of the knot.

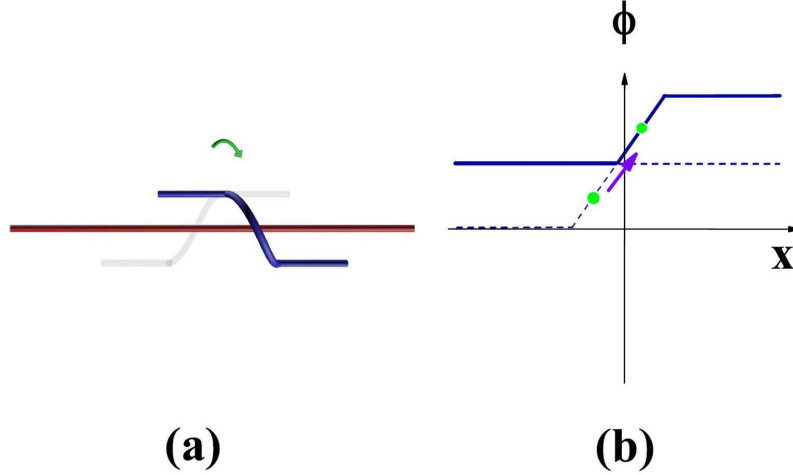


FIG. 4: (a) An illustration for brane-twisting; (b) The relation between phase and coordination for brane-twisting. The green spots denote the average position of the knot.

2. Brane-twisting

The other type of knot motion is brane-twisting. In particular, the brane-twisting cannot be regarded as classical motion in d -dimensional space. Instead, it is regular temporal changing of brane-twisting, a motion of "space background". See the illustration in Fig.4.(a).

For the brane-twisting, the classical-complex-field- d -Brane correspondence is not *valid*. So, we must consider the brane's motion in $d + 2$ dimensional space. We use $z_B|_{x'_1=x_0}(\phi_B(x'_1)) = \hat{U}(\phi_B(x'_1))z_B$ to denote knot at the point

$x'_1 = \vec{e} \cdot \vec{x} = x_0$ where $\phi_B(x'_1)$ is the angle in (x_{d+1}, x_{d+2}) space as

$$\phi_B(x'_1) = \begin{cases} -\frac{\pi}{2}, & x'_1 \in (-\infty, x_0] \\ -\frac{\pi}{2} + \frac{\pi}{a}(x'_1 - x_0), & x'_1 \in (x_0, x_0 + a] \\ \frac{\pi}{2}, & x'_1 \in (x_0 + a, \infty) \end{cases}. \quad (65)$$

For brane-twisting, we have a time-dependent phase angle

$$z_B \rightarrow z_B \exp[i\Delta\phi_B(t, x'_1)]. \quad (66)$$

Because the position of knot is determined by the knot-equation

$$\phi_B(x'_1) + \Delta\phi_B(x'_1, t) = -\frac{\pi}{2} + k_0 \bar{x}'_1 \quad (67)$$

that is

$$\bar{x}'_1 = (x_0 + \frac{a}{2}) + \frac{\Delta\phi_B(x'_1, t)}{k_0}. \quad (68)$$

See the illustration in Fig.4.(b). Due to brane-twisting along given direction \vec{e} , the position of knot is changed. The velocity from brane-twisting is

$$\vec{v}_{\text{twist}} = \frac{1}{k_0} \frac{d[\Delta\phi_B(x'_1, t)]}{dt} \vec{e}. \quad (69)$$

A brane-twisting with angular velocity $\vec{\omega} = \frac{d\phi_B(x'_1, t)}{dt} \vec{e}$ leads to a uniform knot motion in a straight line with fixed velocity

$$\vec{v}_{\text{twist}} = \frac{\vec{\omega}}{k_0}. \quad (70)$$

As a result, for a knot doing motion of both knot-shifting and brane-twisting, the total velocity \vec{v}_{knot} is

$$\vec{v}_{\text{knot}} = \vec{v}_{\text{shift}} + \vec{v}_{\text{twist}}. \quad (71)$$

In particular, the *reality* of a knot depends on its motions: for the case of $\vec{v}_{\text{shift}} \gg \vec{v}_{\text{twist}}$, we have $\vec{v}_{\text{knot}} \simeq \vec{v}_{\text{shift}}$ and then a knot can be regarded as a topological soliton of classical complex field in d -dimensional space, of which the theory is Newton mechanics; for the case of $\vec{v}_{\text{shift}} \ll \vec{v}_{\text{twist}}$, we have $\vec{v}_{\text{knot}} \simeq \vec{v}_{\text{twist}}$ and a knot can be regarded as half-winding of two branes along a given direction \vec{e} , of which the theory cannot be Newton mechanics.

3. Thermalized knots

In above section, it is pointed out that there are two types of knot-motions – knot-shifting and brane-twisting. From the equation $\vec{v} = \vec{v}_{\text{shift}} + \vec{v}_{\text{twist}}$, it looks like that the knot-shifting and brane-twisting play the same role on the motion of a knot. However, the symmetry of the two types of knot-motions is broken by mass-difference $m_{\text{shift}} \neq m_{\text{twist}}$ and thermalization $T_{\text{brane}} \neq 0$. In this part, we consider a d -D linear knot as an entanglement of two *thermalized* d -branes. We may assume the temperature of the branes T_{brane} is equal to that of the $d+2$ dimensional system T_{d+2} , $T_{\text{brane}} = T_{d+2}$.

For the knot-motion from its shifting, the classical-complex-field- d -Brane correspondence is valid and the knot becomes a classical object in d -dimensional space. For this reason, the knot-shifting would be thermalized. The average thermalized energy from knot-shifting is

$$E_{\text{thermal}} = k_B T_{\text{brane}}. \quad (72)$$

At low temperature, the average shifting velocity of a knot on thermalized d -branes is

$$\langle v_{\text{shift}} \rangle = \sqrt{\frac{2k_B T_{\text{brane}}}{d \cdot m_{\text{shift}}}}. \quad (73)$$

In this paper, we focus on the knots on thermalized branes in the limit of $k_B T_{\text{brane}} \gg m_{\text{shift}} c_{5D}^2$ where c_{5D} is light-speed of 5D system. We call it *infinite thermalization condition*. Under the infinite thermalization condition, the average momentum from knot-shifting on thermalized d -branes is given by a universal value $\langle p_{\text{shift}} \rangle = p_0 = \frac{k_B T_{\text{brane}}}{d \cdot c_{5D}}$ that is determined by T_{brane} and 5D light-speed c_{5D} but *independent* of knot shifting-mass m_{shift} .

On the contrary, knot-motion from brane-twisting cannot be thermalized. This is because brane-twisting corresponds to a global motion of the whole system. We may see that *the knot drifts following brane-twisting*. The situation is similar to a drifting boat following water-flow.

As a result, when we consider the knot-motion from brane-twisting, the total average velocity of a knot becomes

$$\begin{aligned} \langle \vec{v}_{\text{knot}} \rangle &= \langle \vec{v}_{\text{shift}} + \vec{v}_{\text{twist}} \rangle \\ &= \vec{v}_{\text{twist}} = \frac{\omega}{k_0} \vec{e}. \end{aligned} \quad (74)$$

The total knot energy is

$$\langle E \rangle = E_{\text{thermal}} + E_{\text{twist}} \quad (75)$$

where

$$E_{\text{twist}} = p_0 v_{\text{twist}} \quad (76)$$

is twisting energy.

A brane-twisting with angular velocity $\frac{d\phi_B(x'_1, t)}{dt} = \omega$ leads to $E_{\text{twist}} = \frac{p_0}{k_0} \omega$. So, we obtain the Planck's energy relationship

$$E_{\text{twist}} = \hbar \omega \quad (77)$$

where the Planck constant is defined by

$$\begin{aligned} \hbar &= \frac{p_0}{k_0} = \frac{\langle p_{\text{shift}} \rangle}{k_0} \\ &= \frac{a \cdot k_B T_{\text{brane}}}{\pi d \cdot c_{5D}}. \end{aligned} \quad (78)$$

The Planck "constant" \hbar will be not a constant any more. It is determined by the temperature of a system T_{brane} and the size of knot's winding region $a = \pi/k_0$.

In the following parts, we focus on the knots on thermalized branes under infinite thermalization condition (or $T_{\text{brane}} \rightarrow \infty$). In this limit, due to thermalization $\langle \vec{v}_{\text{shift}} \rangle = 0$, we have $\langle \vec{v}_{\text{knot}} \rangle = \vec{v}_{\text{twist}}$. As a result, a knot is a "knot" – half-winding of two branes that doesn't obeys classical mechanics. Under infinite thermalization condition, a new mechanics – *knot mechanics* appears that becomes the microscopic origin of quantum mechanics.

D. Knot mechanics – unified theory of quantum mechanics and classical mechanics

1. Fragmentizing knot under infinite thermalization condition

In this part we consider the knots under infinite thermalization condition and focus on the twisting motion for a knot. To describe a thermalized knot that does random shifting motion, we introduce the concept of "fragmentized knot".

A static knot at the point $x'_1 = \vec{e} \cdot \vec{x} = x_0$ is described by $z_B|_{x'_1=x_0}(\phi_B(x'_1)) = \hat{U}(\phi_B(x'_1))z_B$ where $\phi_B(x'_1)$ denotes the angle in (x_{d+1}, x_{d+2}) space as

$$\phi_B(x'_1) = \begin{cases} -\frac{\pi}{2}, & x'_1 \in (-\infty, x_0] \\ -\frac{\pi}{2} + \frac{\pi}{a}(x'_1 - x_0), & x'_1 \in (x_0, x_0 + a] \\ \frac{\pi}{2}, & x'_1 \in (x_0 + a, \infty) \end{cases}. \quad (79)$$

Due to thermalization, the position of a knot $\vec{x}'_1 = \langle x_0 \rangle$ is randomized. We get a uniform distribution of the knot. If we don't consider the brane-twisting, the probability to find a knot at a given point in d -dimensional space is same, i.e., $\rho = \frac{1}{V}$ where V is volume of the system,

$$V = \begin{pmatrix} L_x, & d=1 \\ L_x L_y, & d=2 \\ L_x L_y L_z, & d=3 \end{pmatrix}. \quad (80)$$

$L_x/L_y/L_z$ denotes the size of brane along given direction.

Although the position of a knot is randomized, the number of solution for the knot-equation is conserved. An equivalent representation of a single thermalized knot with random position is *fragmentized knot*.

We firstly split a knot into two pieces, each of them is a half-knot with $\frac{\pi}{2}$ phase-changing. As shown in Fig.5, we split the knot into two pieces. The knot-function of two-piece fragmentized knot is $z_B|_{x'_1=x_0}(\phi_B(x'_1)) = \hat{U}(\phi_B(x'_1))z_B$ where

$$\phi_B(x'_1) = \begin{pmatrix} -\frac{\pi}{2}, x'_1 \in (-\infty, x_0] \\ -\frac{\pi}{2} + \frac{\pi}{a}(x'_1 - x_0), x'_1 \in (x_0, x_0 + \frac{a}{2}] \\ 0, x'_1 \in (x_0 + \frac{a}{2}, x_0 + x'_0] \\ -\frac{\pi}{2} + \frac{\pi}{a}(x'_1 - x_0 - x'_0), x'_1 \in (x_0 + x'_0, x_0 + x'_0 + a] \\ \frac{\pi}{2}, x'_1 \in (x_0 + x'_0 + a, \infty] \end{pmatrix} \quad (81)$$

with a condition of $x'_0 > x_0 + \frac{a}{2}$. It is obvious that for the fragmentized knot, there exists only single knot-solution due to topological condition. And we have 50% probability to find a knot-solution in the region of $x_0 < x'_1 \leq x_0 + \frac{a}{2}$ and 50% probability to find a knot-solution in the region of $x_0 + x'_0 < x'_1 \leq x_0 + x'_0 + a$. We had defined a knot of half-winding to be

$$z_{B,N=1}(\Delta\phi_B(x'_1), x_0) = \hat{U}(\Delta\phi_B = \pi, x_0)z_B, \quad (82)$$

of which the knot is at x_0 . As a result, The knot-function of two-piece fragmentized knot is defined by

$$\begin{aligned} [z_B(\phi_B)]_{\text{fragment}, N=2} &= \hat{U}(\Delta\phi_B = \frac{\pi}{2}, x_0) \\ &\cdot \hat{U}(\Delta\phi_B = \frac{\pi}{2}, x_0 + x'_0)z_B, \end{aligned} \quad (83)$$

of which the two half-knots are at x_0 and $x_0 + x'_0$, respectively.

Similarly, we may split a knot into N pieces, each of which is an identical $\frac{1}{N}$ -knot with $\frac{\pi}{N}$ phase-changing. The knot-function of N -piece knot is

$$\begin{aligned} [z_B(\phi_B)]_{\text{fragment}, N} &= \hat{U}(\Delta\phi_B = \frac{\pi}{N}, (x_0)_1) \\ &\cdot \hat{U}(\Delta\phi_B = \frac{\pi}{N}, (x_0)_2) \dots \\ &\hat{U}(\Delta\phi_B = \frac{\pi}{N}, (x_0)_N)z_B \\ &= \prod_{i=1}^N \hat{U}(\Delta\phi_B = \frac{\pi}{N}, (x_0)_i)z_B \end{aligned} \quad (84)$$

where N identical $\frac{1}{N}$ -knots are at $(x_0)_1, (x_0)_2, \dots, (x_0)_N$, respectively. For simplicity, we assume a periodic distribution of $(x_0)_1, (x_0)_2, \dots, (x_0)_N$. For N -piece fragmentized knot, there also exists only single knot-solution due to topological condition and the knot-number is conserved. We have $\frac{1}{N}$ probability to find a knot-solution in each piece of fragmentized knot.

The knot-number of a knot is given by

$$N_{\text{knot}} = \langle \hat{K} \rangle = \frac{1}{|z_B|^2} \frac{1}{\pi} \int z_{B,N=1}^* \hat{K} z_{B,N=1} d\phi_B \quad (85)$$

where $\hat{K} = -i \frac{d}{d\phi_B}$. To calculate the knot-number, we introduce two types of integrals: one is *on-brane integral* with a sorting of knot-positions in (x_1, x_2, \dots, x_d) space, the other is *out-brane integral* with a sorting of knot-phases (the position in (x_{d+1}, x_{d+2}) space, ϕ_B).

On the one hand, we can label a $\frac{1}{N}$ -knot by its position $(x_0)_{i_x}$ in (x_1, x_2, \dots, x_d) space; on the other hand, we can label a $\frac{1}{N}$ -knot by its position $(x_0)_{i_\phi}$ in (x_{d+1}, x_{d+2}) space. Here i_ϕ denotes a sorting of ordering of phase ϕ_B from small to bigger and i_x denotes a sorting of coordination x with a given order. Each i_ϕ corresponds to an i_x . As a

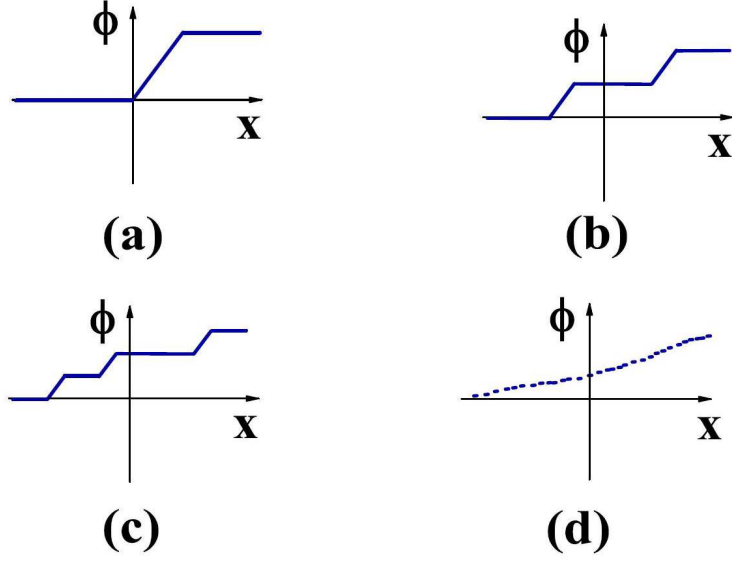


FIG. 5: (a) A knot; (b) A fragmentized knot that is splited into two pieces; (c) A fragmentized knot that is splited three pieces; (d) A fragmentized knot that is splited into infinite pieces. The blue spots denote the fragmentized knot with $N \rightarrow \infty$.

result, there exists an *integral-transformation* by reorganizing the pieces of a knot with different rules as

$$\begin{aligned}
 N_{\text{knot}} &= \frac{1}{|z_B|^2 \pi} \int [z_B(\phi_B)]_{\text{fragment},N}^* \hat{K} [z_B(\phi_B)]_{\text{fragment},N} d\phi_B \\
 &= \frac{1}{|z_B|^2} \sum_{i_\phi=1}^{i_\phi=N \rightarrow \infty} [z_B(\phi_B)]_{\text{fragment},N}^* \hat{K} [z_B(\phi_B)]_{\text{fragment},N} \\
 &= \frac{1}{|z_B|^2} \sum_{i_x=1}^{i_x=N \rightarrow \infty} [z_B(\phi_B)]_{\text{fragment},N}^* \hat{K} [z_B(\phi_B)]_{\text{fragment},N} \\
 &= \frac{1}{|z_B|^2 V} \int [z_B(\phi_B)]_{\text{fragment},N}^* \hat{K} [z_B(\phi_B)]_{\text{fragment},N} dV.
 \end{aligned} \tag{86}$$

For N identical $\frac{1}{N}$ -knots in d -dimensional space, the knot-number is defined by

$$N_{\text{knot}} = \langle \hat{K} \rangle = \frac{1}{|z_B|^2} \frac{1}{\pi} \int [z_B(\phi_B)]_{\text{fragment},N}^* \hat{K} [z_B(\phi_B)]_{\text{fragment},N} d\phi_B \tag{87}$$

where

$$z_B |_{x'_1=x_0} (\Delta\phi_B(x'_1)) = \prod_{i_\phi=1}^{i_\phi=N \rightarrow \infty} \hat{U}(\Delta\phi_B = \frac{\pi}{N}, (x_0)_{i_\phi}) z_B. \tag{88}$$

By transforming an out-brane integral to on-brane integral, the knot-number can be obtained to be

$$N_{\text{knot}} = \langle \hat{K} \rangle = \frac{1}{|z_B|^2} \frac{1}{V} \int [z_B(\phi_B)]_{\text{fragment},N}^* \hat{K} [z_B(\phi_B)]_{\text{fragment},N} dV \tag{89}$$

where

$$z_B |_{x'_1=x_0} (\Delta\phi_B(x'_1)) = \hat{U}(\Delta\phi_B = \frac{\pi}{N}, (x_0)) z_B. \tag{90}$$

As a result, we have

$$\begin{aligned}
N_{\text{knot}} &= \langle \hat{K} \rangle \\
&= \frac{1}{\pi |z_B|^2} \int [z_B(\phi_B)]_{\text{fragment}}^* \hat{K} [z_B(\phi_B)]_{\text{fragment}} d\phi_B \\
&= \frac{1}{|z_B|^2 V} \int z_B^* \{ \hat{U}^*(x) \hat{K} \hat{U}(x) \} z_B dV \\
&= \frac{1}{V} \int dV = 1.
\end{aligned} \tag{91}$$

In the limit of $N \rightarrow \infty$, we have a uniform distribution of the N identical $\frac{1}{N}$ -knot in d -dimensional space. As a result, *it is equivalent to describe a single thermalized knot with random position by fragmented knot with $N \rightarrow \infty$.* Fig.5.(d) is an illustration of a fragmented knot that splits infinite pieces.

2. Wave-function for thermalized knots under waving brane-twisting

With the help of fragmented knot, we study the dynamics of thermalized knots under brane-twisting. Because branes are extended objects, of which the deformation corresponds to classical fluctuation of a complex field $\Phi(\vec{x}, t)$ ($\vec{x} = (x_1, x_2, \dots, x_d)$). From the point view of complex field $\Phi(\vec{x}, t)$, the pattern of its fluctuations corresponds to the wave patterns of brane-twisting. In general, a knot-function of a fragmented knot under waving brane-twisting is defined by

$$\frac{[z_B(\phi_B)]_{\text{fragment}}}{\sqrt{V} |z_B|}. \tag{92}$$

We point out that for an arbitrary waving pattern of brane-twisting, *the knot-function of a fragmented knot under waving brane-twisting plays the role of the wave-function in quantum mechanics* as

$$\frac{[z_B(\phi_B)]_{\text{fragment}}}{\sqrt{V} |z_B|} \implies \psi(\vec{x}, t) = \sqrt{\Omega(\vec{x}, t)} e^{i\phi(\vec{x}, t)}, \tag{93}$$

and

$$\begin{aligned}
\Delta\phi_B(\vec{x}, t) &\implies \phi(\vec{x}, t), \\
\rho_{\text{knot}} &\implies \Omega(\vec{x}, t)
\end{aligned} \tag{94}$$

where the knot-function $\frac{[z_B(\phi_B)]_{\text{fragment}}}{\sqrt{V} |z_B|}$ of knot mechanics corresponds to the wave-function $\psi(\vec{x}, t)$ in quantum mechanics, the angle from brane-twisting $\Delta\phi_B(\vec{x}, t)$ of knot mechanics corresponds to the quantum phase angle $\phi(\vec{x}, t)$ of wave-function and the knot density $\rho_{\text{knot}} = \frac{N_{\text{knot}}}{\Delta V} = \langle \frac{\hat{K}}{\Delta V} \rangle$ of knot mechanics corresponds to the particle probability $\Omega(\vec{x}, t)$ of the wave-function.

We then prove that the wave-function $\psi(\vec{x}, t) = \sqrt{\Omega(\vec{x}, t)} e^{i\phi(\vec{x}, t)}$ (we call knot-function) is *the function of probability wave* for knots (elementary particles). Because a knot is phase-changing, the knot density is defined by $\rho_{\text{knot}} = \frac{N_{\text{knot}}}{\Delta V} =$

$\left\langle \frac{\hat{K}}{\Delta V} \right\rangle$ that is

$$\begin{aligned}
\rho_{\text{knot}} &= \frac{1}{\Delta V |z_B|^2 \pi} [z_B(\vec{x}, t)]_{\text{fragment}}^* \hat{K} [z_B(\vec{x}, t)]_{\text{fragment}} d\phi_B \\
&= \frac{1}{\Delta V |z_B|^2} \sum_{i_\phi} [z_B(t, (x_0)_{i_\phi})]_{\text{fragment}}^* \hat{K} [z_B(t, (x_0)_{i_\phi})]_{\text{fragment}} \\
&= \frac{1}{\Delta V |z_B|^2} \sum_{i_x} [z_B(t, (x_0)_{i_x})]_{\text{fragment}}^* \hat{K} [z_B(t, (x_0)_{i_x})]_{\text{fragment}} \\
&= \frac{1}{\Delta V |z_B|^2} [z_B(\vec{x}, t)]_{\text{fragment}}^* \hat{K} [z_B(\vec{x}, t)]_{\text{fragment}} dV \\
&= \frac{1}{\Delta V} \psi^*(\vec{x}, t) (-i \frac{d}{d\phi}) \psi(\vec{x}, t) dV \\
&= \psi^*(\vec{x}, t) \psi(\vec{x}, t) = \Omega(\vec{x}, t).
\end{aligned} \tag{95}$$

As a result, $\psi(\vec{x}, t)$ indeed plays the role of wave-function in quantum mechanics, $\Omega(\vec{x}, t)$ is the particle probability (the knot density ρ_{knot}) and $\phi(\vec{x}, t)$ is the phase (the angle in extra space). For a single knot, we have normalization condition $\int \Omega(\vec{x}, t) dV = 1$.

The physical reason of *the probability wave for knots (elementary particles) comes from the phase-change nature of a knot*. For a thermalized knot, *the phase-change from waving brane-twisting and that from a knot cannot be distinguished*. As a result, the probability to find a knot becomes the probability to find a phase-change. This is reason why $\psi(\vec{x}, t) = \sqrt{\Omega(\vec{x}, t)} e^{i\phi(\vec{x}, t)}$ is probability wave.

The relation $\rho_{\text{knot}} = \Omega(\vec{x}, t)$ can be proved by another approach. From the definition of knot-number operator $\hat{N}_{\text{knot}} = \hat{K}_{\text{knot}} = -i \frac{d}{d\phi}$, we have commutation relation between knot-number operator and quantum phase

$$[\phi, \hat{N}_{\text{knot}}] = \left[\phi, -i \frac{d}{d\phi} \right] = i. \tag{96}$$

The commutation relation between knot-number operator and quantum phase leads to a phase operator, $\hat{\phi} = -i \frac{d}{dN_{\text{knot}}}$ that gives

$$[\hat{\phi}, N_{\text{knot}}] = i \tag{97}$$

in knot-number representation. For this reason, the wave-function plays the role of a generating operation of a knot-number

$$\begin{aligned}
&\hat{\psi}^*(\vec{x}, t) \cdot N_{\text{knot}, 0} \cdot \hat{\psi}(\vec{x}, t) \\
&= \sqrt{\Omega(\vec{x}, t)} e^{-i\hat{\phi}(\vec{x}, t)} \cdot N_{\text{knot}, 0} \cdot \sqrt{\Omega(\vec{x}, t)} e^{i\hat{\phi}(\vec{x}, t)} \\
&= \Omega(\vec{x}, t) (N_{\text{knot}, 0} + \delta N_{\text{knot}})
\end{aligned} \tag{98}$$

where $\delta N_{\text{knot}} = 1$. If we define the bare knot-number is zero for two parallel flat d -brane, i.e., $N_{\text{knot}, 0} = 0$, the knot density $\rho_{\text{knot}} = |\psi(\vec{x}, t)|^2 = \Omega(\vec{x}, t)$ is the probability to find a knot.

Thus, the quantum state is invariant under a global gauge transformation, i.e.,

$$\begin{aligned}
\psi(\vec{x}, t) &\rightarrow \psi'(\vec{x}, t) = \psi(\vec{x}, t) e^{i\phi_0} \\
&= \sqrt{\Omega(\vec{x}, t)} e^{i\phi(\vec{x}, t)} e^{i\phi_0}
\end{aligned} \tag{99}$$

where ϕ_0 is constant. Such invariant comes from a global rotation symmetry in (x_{d+1}, x_{d+2}) space.

3. Superposition principle

Without considering brane-twisting, the thermalized knot has uniform distribution and we may use static fragmented knot with $N \rightarrow \infty$ to characterize its behavior. We define the knot-function of static fragmented knot as

$$[z_B(\phi_B)]_{\text{fragment}} = \left[\hat{U}(\phi_B) \right]_{\text{fragment}} z_B, \tag{100}$$

of which there is no information of the position of a knot. Because the probability to find a static fragmentized knot is $\rho_{\text{knot}} = \frac{1}{V}$, we define normalized knot-function of static fragmentized knot,

$$\frac{[z_B(\phi_B)]_{\text{fragment}}}{\sqrt{V} |z_B|} \Rightarrow \sqrt{\rho_{\text{knot}}} = \frac{1}{\sqrt{V}}. \quad (101)$$

Next, we consider a special type of brane-twisting – *plane-waving brane-twisting*, of which the knot-function of a fragmentized knot is

$$\begin{aligned} \frac{[z_B(\phi_B)]_{\text{fragment}}}{\sqrt{V} |z_B|} &= \frac{1}{\sqrt{V} |z_B|} \left[\hat{U}(\phi_B) \right]_{\text{fragment}} z_B(\vec{x}, t) \\ &= \frac{1}{\sqrt{V} |z_B|} [z_B(\phi_B)]_{\text{fragment}} \exp[i\Delta\phi_B(\vec{x}, t)] z_B \\ &= \frac{1}{\sqrt{V}} \exp[i\Delta\phi_B(\vec{x}, t)] \end{aligned} \quad (102)$$

where $\Delta\phi_B(\vec{x}, t) \rightarrow \phi(\vec{x}, t) = \omega t - \vec{k} \cdot \vec{x}$ and \vec{k} is the wave-vector of waving brane-twisting. The knot-function of a fragmentized knot under waving brane-twisting is just the wave-function of a plane wave in quantum mechanics,

$$\begin{aligned} \frac{[z_B(\phi_B)]_{\text{fragment}}}{\sqrt{V} |z_B|} &\Rightarrow \psi(\vec{x}, t) \\ &= \frac{1}{\sqrt{V}} e^{i\Delta\phi_B(\vec{x}, t)} \\ &= \frac{1}{\sqrt{V}} e^{i(\omega t - \vec{k} \cdot \vec{x})}. \end{aligned} \quad (103)$$

From the fact of superposition principle of wave, a generalized knot-function can be

$$\begin{aligned} \frac{[z_B(\phi_B)]_{\text{fragment}}}{\sqrt{V} |z_B|} &\Rightarrow \psi(\vec{x}, t) = \sqrt{\Omega(\vec{x}, t)} e^{i\phi(\vec{x}, t)} \\ &= \sum_{\vec{k}} c_{\vec{k}} \exp(i\omega(\vec{k}) \cdot t - i\vec{k} \cdot \vec{x}) \end{aligned} \quad (104)$$

where $c_{\vec{k}}$ is the amplitude of given plane wave \vec{k} . Here we expand the knot-function (wave-function) $\psi(\vec{x}, t)$ by plane-waves. The results illustrate superposition principle.

As a result, the knot-functions of a fragmentized knot under an arbitrary waving brane-twisting have the superposition effect of different patterns of waving brane-twisting that leads to the superposition of wave-function in quantum mechanics.

4. Energy/momentum operator

For a plane wave, $\psi(\vec{x}, t) = \frac{1}{\sqrt{V}} \exp(i\omega t - i\vec{k} \cdot \vec{x})$, the energy of a knot with twisting motion is

$$E_{\text{twist}} = \hbar\omega. \quad (105)$$

We then define the momentum of a knot under waving brane-twisting

$$\vec{p}_{\text{twist}} = \hbar\vec{k} \quad (106)$$

where \vec{k} is wave vector of brane-twisting. In addition, we show the definition of energy and momentum in knot mechanics in the following equations,

$$\begin{aligned} \text{Energy} &\Rightarrow \text{Time-dependence of phase changing,} \\ \text{Momentum} &\Rightarrow \text{Spatial-dependence of phase changing.} \end{aligned} \quad (107)$$

As a result, we have

$$\psi(\vec{x}, t) = \frac{1}{\sqrt{V}} \exp\left(\frac{iE_{\text{twist}}t - i\vec{p}_{\text{twist}} \cdot \vec{x}}{\hbar}\right). \quad (108)$$

From the fact of superposition principle of wave, a generalized knot-function can be

$$\begin{aligned} \psi(\vec{x}, t) &= \sqrt{\Omega(\vec{x}, t)} e^{i\phi(\vec{x}, t)} \\ &= \sum_{\vec{p}} c_{\vec{p}} \exp\left(\frac{iE_{\text{twist}}t - i\vec{p}_{\text{twist}} \cdot \vec{x}}{\hbar}\right). \end{aligned} \quad (109)$$

Next, we calculate the expect values of the energy $\langle E \rangle$ and the momentum $\langle \vec{p} \rangle$, respectively.

From the wave-function $\psi(\vec{x}, t) = \sum_{\vec{p}} c_{\vec{p}} \exp\left(\frac{i(E_{\text{twist}}t - \vec{p}_{\text{twist}} \cdot \vec{x})}{\hbar}\right)$, we have

$$\begin{aligned} \langle E \rangle &= \int E_{\text{twist}} \Omega(x) dV = \int \psi^*(\vec{x}, t) E_{\text{twist}} \psi(\vec{x}, t) dV \\ &= \int \left[\sum_{\vec{p}} c_{\vec{p}}^* \exp\left(\frac{-i(E_{\text{twist}}t - \vec{p}_{\text{twist}} \cdot \vec{x})}{\hbar}\right) \right] \\ &\quad E \left[\sum_{\vec{p}'} c_{\vec{p}'} \exp\left(\frac{i(E_{\text{twist}}t - \vec{p}'_{\text{twist}} \cdot \vec{x})}{\hbar}\right) \right] dV \\ &= \int \left[\sum_{\vec{p}} c_{\vec{p}}^* \exp\left(\frac{-i(E_{\text{twist}}t - \vec{p}_{\text{twist}} \cdot \vec{x})}{\hbar}\right) \right] \\ &\quad \left(-i \frac{d}{dt} \right) \left[\sum_{\vec{p}'} c_{\vec{p}'} \exp\left(\frac{i(E_{\text{twist}}t - \vec{p}'_{\text{twist}} \cdot \vec{x})}{\hbar}\right) \right] dV \\ &= \int \psi^*(\vec{x}, t) \left(-i \frac{d}{dt} \right) \psi(\vec{x}, t) dV. \end{aligned} \quad (110)$$

This result indicates that the energy from twisting motion becomes operator

$$E_{\text{twist}} \rightarrow \hat{E}_{\text{twist}} = -i\hbar \frac{d}{dt}. \quad (111)$$

Using the similar approach, we derive

$$\begin{aligned} \langle \vec{p} \rangle &= \int \vec{p} \Omega(\vec{x}, t) dV = \int \psi^*(\vec{x}, t) \vec{p} \psi(\vec{x}, t) dV \\ &= \int \psi^*(\vec{x}, t) \left(i \frac{d}{d\vec{x}} \right) \psi(\vec{x}, t) dV. \end{aligned} \quad (112)$$

As a result, the momentum from twisting motion also becomes operators

$$\vec{p}_{\text{twist}} \rightarrow \hat{p}_{\text{twist}} = i\hbar \frac{d}{d\vec{x}}. \quad (113)$$

5. The energy-momentum relationship and Schrödinger equation

The energy-momentum relationship $E_{\text{twist}} = H(\vec{p}_{\text{twist}})$ determines the equation of motion for knot-function. However, we have to give an assumption on the formulation of the energy-momentum relationship $E_{\text{twist}} = H(\vec{p}_{\text{twist}})$. If we assume

$$E_{\text{twist}} = \frac{\vec{p}_{\text{twist}}^2}{2m_{\text{twist}}}, \quad (114)$$

the Schrödinger equation for wave-function is obtained[16]

$$\hat{E}_{\text{twist}} \psi(\vec{x}, t) = \frac{\hat{p}_{\text{twist}}^2}{2m_{\text{twist}}} \psi(\vec{x}, t) \quad (115)$$

or

$$-i\hbar \frac{d\psi(\vec{x}, t)}{dt} = \frac{\hat{p}_{\text{twist}}^2}{2m_{\text{twist}}} \psi(\vec{x}, t). \quad (116)$$

For the eigenstate with eigenvalue E , the wave-function is given by

$$\psi(\vec{x}, t) = f(\vec{x}) \exp\left(\frac{iE_{\text{twist}} t}{\hbar}\right) \quad (117)$$

where $f(\vec{x})$ is spacial function. This state corresponds to a brane-twisting with fixed twisting angular velocity, $\omega = \frac{E_{\text{twist}}}{\hbar}$. That means the excitations of the quantum state must have the quantized energy, $\Delta E = nE_{\text{twist}} = n\hbar\omega$ where n is a positive integer number.

6. Second quantization for many-body knots and spin-statistic relation

We can generalize the knot-mechanics to a many-knot system. We may introduce N -wave-function to describe twisting motions of N -knots, $\Psi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N)$. The situation is just that in a many-body quantum mechanics. An important feature is the statistics. In quantum mechanics, an assumption is spin-statistics of fermions. An electron is a fermion and its spin is $1/2$. The wave-function of a system of identical spin- $1/2$ particles changes sign when two particles are swapped. Particles with wave-functions antisymmetric under exchange are called fermions. To distinguish the statistics for the knots, we consider two knots.

We show that *the wave-functions antisymmetric by braiding knots is due to π -phase changing nature of knots*. See the illustration in Fig.6. When we braid two knots, the angle in extra space (that is just the phase of wave-function) is π . According to the definition of knots, we have two static knots and get knot-functions

$$\Psi(\vec{x}, \vec{x}') = \hat{U}(\phi'(\vec{x}')) \cdot \hat{U}(\phi(\vec{x})) z_B. \quad (118)$$

Due to

$$\hat{U}^*(\phi(\vec{x})) \cdot \hat{U}(\phi'(\vec{x}')) \cdot \hat{U}(\phi(\vec{x})) = -\hat{U}(\phi'(\vec{x}')), \quad (119)$$

after exchanging two knots, we get

$$\begin{aligned} \Psi(\vec{x}', \vec{x}) &= [\hat{U}(\phi'(\vec{x}')) \cdot \hat{U}(\phi(\vec{x}))] z_B \\ &\rightarrow z(\vec{x}, \vec{x}') = [\hat{U}(\phi(\vec{x})) \cdot \hat{U}(\phi'(\vec{x}'))] z_B \\ &= [\hat{U}(\phi'(\vec{x}')) \cdot \hat{U}(\phi(\vec{x}))] z'_B \\ &= -\Psi(\vec{x}, \vec{x}') \end{aligned} \quad (120)$$

where $z'_B = z_B e^{i\pi}$.

As a result, in second quantization representation, we introduce fermionic operator $c_1^\dagger(\vec{x})$, $c_2^\dagger(\vec{x})$ to describe knot braiding

$$c_1^\dagger(\vec{x}) \implies \hat{U}(\phi(\vec{x})), \quad c_2^\dagger(\vec{x}) \implies \hat{U}(\phi'(\vec{x}')). \quad (121)$$

The flat d -brane without entanglement corresponds to the vacuum state $|0\rangle$

$$z_B \implies |0\rangle. \quad (122)$$

Then after exchanging two identical particles, the many-body wave-function

$$\Psi_{\text{initial}}(\vec{x}', \vec{x}) = c^\dagger(\vec{x}) c^\dagger(\vec{x}') |0\rangle \quad (123)$$

changes by a phase to be

$$\begin{aligned} \Psi_{\text{initial}}(\vec{x}', \vec{x}) &\rightarrow \Psi_{\text{final}}(\vec{x}', \vec{x}) \\ &= e^{i\pi} c^\dagger(\vec{x}') c^\dagger(\vec{x}) |0\rangle. \end{aligned} \quad (124)$$

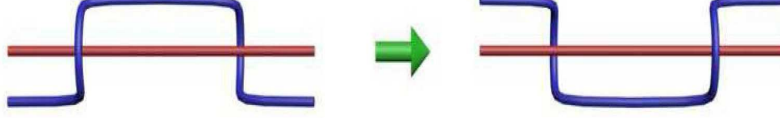


FIG. 6: Illustration of anticommutation relation by braiding two knots

Next, an important concept is spin degrees of freedom. We had shown that there exist two types of entanglement patterns that are described by

$$\mathbf{z}(x - x_0) = \begin{pmatrix} b_A \\ b_B \end{pmatrix} = \frac{1}{r_0} \begin{pmatrix} \Phi_{1,B}(x - x_0) \\ \Phi_{2,B}(x - x_0) \end{pmatrix}. \quad (125)$$

For knots, we identify b_A the knot-function of a clockwise winding knot to be the wave-function of spin- \uparrow , b_B the knot-function of an anti-clockwise winding knot to be the wave-function of spin- \downarrow . See the illustration in Fig.7. As a result, a knot is an object with spin-1/2. To characterize the two knots, we introduce the representation via fermionic operator

$$\mathbf{z}(\vec{x} - \vec{x}_0) = \begin{pmatrix} b_A \\ b_B \end{pmatrix} \implies \psi(\vec{x}, t) = \begin{pmatrix} \psi_{\uparrow}(\vec{x}, t) \\ \psi_{\downarrow}(\vec{x}, t) \end{pmatrix} \quad (126)$$

or the two operators $c(\vec{x}, t) = \begin{pmatrix} c_{\uparrow}(\vec{x}, t) \\ c_{\downarrow}(\vec{x}, t) \end{pmatrix}$. We identify $c_{\uparrow}^{\dagger}(\vec{x}, t)$ ($c_{\uparrow}(\vec{x}, t)$) the level (or annihilation) operator of a clockwise winding knot or a spin- \uparrow knot; $c_{\downarrow}^{\dagger}(\vec{x}, t)$ ($c_{\downarrow}(\vec{x}, t)$) the level (or annihilation) operator of an anti-clockwise winding knot or a spin- \downarrow knot.

An important property of spin degrees of freedom is inversion of spin-component under time-reversal operation. We define a time-reversal operator \hat{T} , by which we have

$$\hat{T} \begin{pmatrix} b_A \\ b_B \end{pmatrix} = \begin{pmatrix} b_B \\ b_A \end{pmatrix} \implies \hat{T} \begin{pmatrix} c_{\uparrow}(\vec{x}, t) \\ c_{\downarrow}(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} c_{\downarrow}(\vec{x}, t) \\ c_{\uparrow}(\vec{x}, t) \end{pmatrix}. \quad (127)$$

In physics, above equation means under time-reversal operation, a clockwise winding knot (spin- \uparrow particle) turns into an anti-clockwise winding knot (spin- \downarrow particle).

Finally, we discuss the spin degrees of freedom under spacial rotation operation \hat{S} . For a 1D knot, there is no spacial rotation operation. For a 2D knot, we can define a 2D spacial rotation operation $\hat{S}(\varphi)$. We can rotate the 2D knot-function 90° , $Z \rightarrow Ze^{i\frac{\pi}{2}}$ i.e. x-axis to y-axis and get

$$\begin{pmatrix} \ln\left[\frac{z(Z)}{r_0}\right] \\ \ln\left[\frac{\bar{z}(\bar{Z})}{r_0}\right] \end{pmatrix} \rightarrow \begin{pmatrix} \ln\left[\frac{z(Ze^{i\frac{\pi}{2}})}{r_0}\right] - \frac{\pi}{2} \\ \ln\left[\frac{\bar{z}(\bar{Z}e^{i\frac{\pi}{2}})}{r_0}\right] + \frac{\pi}{2} \end{pmatrix} \quad (128)$$

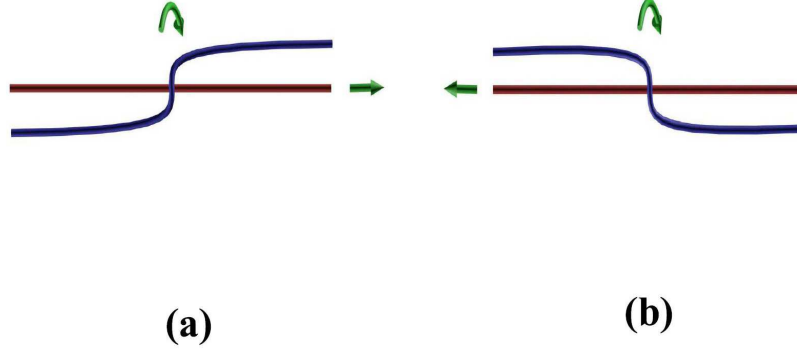


FIG. 7: Illustration of spin: (a) a spin-up knot, that moves forwards during brane-twisting clockwise; (b) a spin-down knot, that moves backwards during Brane-twisting clockwise.

or

$$\mathbf{z}(\vec{x} - \vec{x}_0) = \begin{pmatrix} b_A \\ b_B \end{pmatrix} = \frac{1}{r_0} \begin{pmatrix} \Phi_{1,B}(x - x_0) \\ \Phi_{2,B}(x - x_0) \end{pmatrix} \quad (129)$$

$$\rightarrow \begin{pmatrix} b_A \cdot e^{-i\frac{\pi}{2}} \\ b_B \cdot e^{i\frac{\pi}{2}} \end{pmatrix}.$$

This rotation operation corresponds to $\hat{R} = \exp(i\frac{\pi}{2}\sigma_z)$ that leads to

$$\begin{pmatrix} c_\uparrow(\vec{x}, t) \\ c_\downarrow(\vec{x}, t) \end{pmatrix} = \hat{R} \begin{pmatrix} c_\uparrow(\vec{x}, t) \\ c_\downarrow(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\pi}{2}} c_\downarrow(\vec{x}, t) \\ e^{i\frac{\pi}{2}} c_\uparrow(\vec{x}, t) \end{pmatrix}. \quad (130)$$

As a result, the two components of 2D knot are 2D spinor.

It is obvious that the 3D knot is really a "magnetic" monopole that shows a hedgehog behavior at $(0, 0, 0)$. By defining $O(3)$ rotor from two-component spinor, $n^a(x)\sigma^a = \bar{\mathbf{z}}\sigma\mathbf{z}$ ($a = 1, 2, 3$), we have a 3D knot-solution of the winding 3-brane out of the winding region as

$$\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|}, \quad r \geq a. \quad (131)$$

The statistical translation in the presence of monopoles had been discovered [17]: in broken $SU(2)$ gauge theory, isospin degrees of freedom can be converted into spin degrees of freedom in the field of a magnetic monopole. If an object has half integer isospin, a fermion with half integer spin emerges. As a result, the 3D linear knot has rotation symmetry in 3D space and the rotation operation corresponds to

$$\hat{R} = \exp(-i\varphi\sigma_z) \exp(-i\theta\sigma_y) \exp(i\varphi\sigma_z). \quad (132)$$

From above discussion, the two types of entanglement patterns play the role of spin degrees of freedom.

Finally, we discuss the underline physics of *Pauli exclusion principle*. In 1925, W. Pauli pointed out a principle: "the quantum mechanical principle that states that two identical fermions (particles with half-integer spin) cannot occupy the same quantum state simultaneously". In knot mechanics, the Pauli exclusion principle comes from the quantum states classified by different topological numbers, $\mathcal{N}_{\text{dD-knot}} = \pm 1$. According to the complex-field-brane correspondence, knot becomes topological defect in classical complex field: 1D knot is a π -domain wall; 2D knot is a 2π -flux; 3D knot is a Dirac monopole. When there exist N knots, the total topological number of the system is $\mathcal{N}_{\text{dD-knot}} = \pm N$. As a result, the states with different numbers of knots must be different quantum states. This is topological origin of Pauli exclusion principle.

7. Path integral formulation

In 1948, Feynman derived path integral formulation of quantum mechanics, based on the fact that the propagator can be written as a sum over all possible paths between the initial and final points.

With the help of the path integral, people can calculate the probability amplitude $K(x', t_f; x, t_i)$ from an initial position x at time $t = t_i$ (that is described by a state $|t_i, x\rangle$) to position x' at a later time $t = t_f$ ($|t_f, x'\rangle$). Then the initial state is $|x\rangle$. Letting the state evolve in time and projecting on the state $|x'\rangle$,

$$\begin{aligned} K(x', t_f; x, t_i) &= \langle t_f, x' | t_i, x \rangle = \sum_n e^{iS_n/\hbar} \\ &= \int \mathcal{D}p(t) \mathcal{D}x(t) e^{iS/\hbar} \end{aligned} \quad (133)$$

where $S = \int [p\dot{x} - H(p, x)]dt$. Each path contributes $e^{iS_n/\hbar}$ where S_n is the n -th classical action i.e., $\sum_n e^{iS_n/\hbar}$.

Let us discuss the implication of path-integral formulation in knot mechanics. We firstly consider a knot at the position (x_i, ϕ_0) and time $t = t_i$ where x_i denotes the original position on brane and ϕ_0 the position out brane. To calculate the probability amplitude $K(x', t_f; x, t_i) = \langle t_f, x' | t_i, x \rangle$, we consider fragmentized knot by splitting a knot into N parts ($N \rightarrow \infty$). For each $\frac{1}{N}$ -knot, the classical action is S_n where n is an arbitrary possible classical path from (x_i, ϕ_0) to (x_f, ϕ_n) within time $T = t_f - t_i$. In particular, the position out of the brane ϕ may be difference for a knot moving along different classical paths. The phase changing of a possible classical path is given by $\Delta\phi = \phi_n - \phi_0 = \frac{S_n}{\hbar}$. As a result, we have the knot-function as $e^{iS_n/\hbar}$. We summarize the contribution from all $\frac{1}{N}$ -knots and get the final knot-function as

$$\sum_n e^{i\Delta\phi_n} = \sum_n e^{iS_n/\hbar} \quad (134)$$

that is just the Feynman's path integral formulation.

For the case with many-knot, we may introduce N -wave-function to describe twisting motions of N -knots, $\Psi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N)$. With the help of particle operators, we define the total Hamiltonian of the system with multi-knot as

$$\hat{\mathcal{H}} = \int d^3x [\psi^\dagger(\vec{x}) \hat{H} \psi(\vec{x})] \quad (135)$$

where \hat{H} is the Hamiltonian operator of single knot and $\psi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_p c_p \exp(i\vec{p} \cdot \vec{x}/\hbar)$ ($c_p = \begin{pmatrix} c_{\uparrow, p} \\ c_{\downarrow, p} \end{pmatrix}$). As a result, we have

$$\hat{\mathcal{H}} = \sum_k H(\vec{p}) c_p^\dagger c_p = \sum_k H(\vec{p}) \hat{n}_p \quad (136)$$

where $\hat{n}_p = c_p^\dagger c_p$. From the definition of total Hamiltonian $\hat{\mathcal{H}}$ means that for a given mode (brane-twisting pattern) there are n_p knots.

Now, we consider the path integral formulation of multi-knot. The probability amplitude $K(\vec{x}'_M, \dots, \vec{x}'_2, \vec{x}'_1, t_f; \vec{x}_M, \dots, \vec{x}_2, \vec{x}_1, t_i) = \langle t_f, \vec{x}'_M, \dots, \vec{x}'_2, \vec{x}'_1 | t_i, \vec{x}_M, \dots, \vec{x}_2, \vec{x}_1 \rangle$ becomes a multi-variable function where \vec{x}'_j and \vec{x}_j denote the final position and initial position of j -th knot, respectively. For a multi-knot system, quantum processes are described by

$$\begin{aligned} &K(\vec{x}'_M, \dots, \vec{x}'_2, \vec{x}'_1, t_f; \vec{x}_M, \dots, \vec{x}_2, \vec{x}_1, t_i) \\ &= \langle t_f, \vec{x}'_M, \dots, \vec{x}'_2, \vec{x}'_1 | t_i, \vec{x}_M, \dots, \vec{x}_2, \vec{x}_1 \rangle \\ &= \prod_j \sum_n e^{i\Delta\phi_{j,n}} = \prod_j \sum_n e^{iS_{j,n}/\hbar} = \sum_n e^{i \sum_j S_{j,n}/\hbar} \\ &= \prod_p \psi_p^\dagger(\vec{x}, t) \psi_p(\vec{x}, t) e^{iS_p/\hbar} = \int \mathcal{D}\psi^\dagger(\vec{x}, t) \mathcal{D}\psi(\vec{x}, t) e^{iS/\hbar} \end{aligned} \quad (137)$$

where $S = \sum_{\omega, \vec{p}} S_{\omega, \vec{p}} = \int \mathcal{L} dt d^3x$ with $S_{\omega, \vec{p}} = \psi_p^\dagger(i\hbar\omega(\vec{p}) - H(\vec{p}))\psi_p$ and $\mathcal{L} = i\psi^\dagger \partial_t \psi - \hat{\mathcal{H}}$.

E. Principles of knot mechanics

The key principle of quantum mechanics is the complementarity principle that was formulated by N. Bohr. In quantum mechanics, quantum objects have complementary properties which cannot be measured accurately at the same time: the more accurately one property is measured, the less accurately the complementary property is measured. For example, position and momentum of a particle have complementary properties. In this section, we show the key principles in knot mechanics. There are two principles of knot mechanics – *matter-motion complementarity principle* and *quantum-classical contradictoriness*. In the end, we discuss the measurement theory in knot mechanics.

F. Matter-motion complementarity principle

In knot mechanics, complementarity principle comes from complementarity property of knots. A knot is an entanglement two d -branes. On the one hand, a knot is π -phase changing – a *sharp, time-independent, topological* phase changing, $\Delta\phi = \phi_B(x'_1 \rightarrow \infty) - \phi_B(x'_1 \rightarrow -\infty) = \pm\pi$. To characterize the phase changing property of a knot, we had introduced knot-number operator $\hat{K} = -i\frac{d}{d\phi_B}$; on the other hand, a knot has a phase angle, ϕ_0 . Knot mechanics describes the dynamics of *smooth, slow, non-topological* phase changing, $\Delta\phi(t, x'_1)$.

To generate a knot we had introduced knot-operation $\hat{U}(\phi_B(x'_1)) = \exp[i\phi_B(x'_1) \cdot \hat{K}]$ on d -brane-B along the direction \vec{e} where $\phi_B(x'_1)$ is a given function of a linear knot for a π -phase changing. Under infinite thermalization condition, the π -phase changing becomes fragmentized and may uniform spread in the 3D space as

$$[z_B(\phi_B)]_{\text{fragment}} = \left[\hat{U}(\phi_B(x'_1)) \right]_{\text{fragment}} z_B. \quad (138)$$

For a knot, phase changing is sharp, time-independent as $\phi_B(x'_1) = k_0 x'_1$ in the winding region of $x_0 < x'_1 \leq x_0 + a$.

Using similar approach, we may "generate" a particular waving brane-twisting pattern by a brane-twisting operation

$$\hat{U}_w(\phi_B(t, x'_1)) = \exp[i\phi_{w,B}(t, x'_1) \cdot \hat{K}]. \quad (139)$$

For example, for a plane-wave, $\psi(\vec{x}, t) = \frac{1}{\sqrt{V}} \exp[\frac{iE_{\text{twist}}t - i\vec{p}_{\text{twist}} \cdot \vec{x}}{\hbar}]$ we have

$$\phi_{w,B}(t, x'_1) = E_{\text{twist}}t - \vec{p}_{\text{twist}} \cdot \vec{x} \quad (140)$$

and get the knot-function to be

$$\begin{aligned} [z_B(\phi_B)]_{\text{fragment}} &\rightarrow [z'_B(\phi_B)]_{\text{fragment}} \\ &= \hat{U}_w(\phi_B(t, x'_1)) \left[\hat{U}(\phi_B(x'_1)) \right]_{\text{fragment}} z_B. \end{aligned} \quad (141)$$

For knot motion from waving brane-twisting, phase changing is smooth as

$$\frac{\Delta\phi(t, x'_1)}{\Delta x'_1} = p_{\text{twist}}/\hbar \ll k_0.$$

In particular, there exists a commutation relationship

$$\begin{aligned} \hat{U}_w(\phi_B(t, x'_1)) \left[\hat{U}(\phi_B(x'_1)) \right]_{\text{fragment}} \\ = \left[\hat{U}(\phi_B(x'_1)) \right]_{\text{fragment}} \hat{U}_w(\phi_B(t, x'_1)). \end{aligned} \quad (142)$$

This commutation relationship indicates an important feature of knot mechanics – *matter-motion complementarity principle*. That means in knot mechanics we may *unify matter (π -phase changing) and its motion (smooth phase changing) into a phenomenon – phase changing (entanglement between branes)*. Before observing, we *cannot* distinguish a fragmentized knot from its motion. So, we can superpose a fragmentized knot on its motion.

Based on matter-motion complementarity principle, we explain wave-particle duality in quantum mechanics. Wave-particle duality is the fact that elementary particles exhibit both particle-like behavior and wave-like behavior. As Einstein wrote: "It seems as though we must use sometimes the one theory and sometimes the other, while at times we may use either. We are faced with a new kind of difficulty. We have two contradictory pictures of reality; separately neither of them fully explains the phenomena of light, but together they do". Here, we point out that wave-particle

duality of quantum particles is really "*matter-motion duality*" that is property of matter-motion complementarity principle. On the one hand, a knot is π -phase changing – a sharp, time-independent, topological phase changing. A knot becomes a "point" after full-projection. Thus it shows particle-like behavior; on the other hand, the dynamics of smooth, slow, non-topological phase changing of a knot shows wave-like behavior that is characterized by wave-functions. We show the correspondence between wave-particle duality and matter-motion duality in the following equations,

$$\text{Wave-particle duality} \iff \text{matter-motion duality}, \quad (143)$$

and

$$\begin{aligned} \text{Particle-like behavior} &\iff \text{matter (sharp, time-independent,} \\ &\quad \text{topological phase changing),} \\ \text{Wave-like behavior} &\iff \text{motion (smooth, slow,} \\ &\quad \text{non-topological phase changing).} \end{aligned}$$

1. Quantum-classical contradictoriness

We had shown that there are two types of motions of a knot: knot-shifting and brane-twisting. The reality of a knot depends on its motions: for the case of $\vec{v}_{\text{shift}} \gg \vec{v}_{\text{twist}}$, a knot can be regarded as a topological soliton of classical complex field in d -dimensional space, of which the theory is Newton mechanics; for the case of $\vec{v}_{\text{shift}} \ll \vec{v}_{\text{twist}}$, a knot can be regarded as a "knot" – half-winding of two branes, of which the theory is quantum mechanics. Therefore, we introduce the concept of *quantum-classical contradictoriness* and say that the answer of "what is the matter" is determined by the answer of "how matter moves". We show it by the following equation,

$$\text{A knot} = \text{Classical soliton/Quantum particle}. \quad (144)$$

We consider a heavy object (for example, the mass of the object is $M \gg m_{\text{twist}}$) that consists of a lot of knots. In quantum mechanics, the heavy object also does quantum motion and its behavior obeys the Schrödinger equation. However, the situation changes for the case in knot-mechanics.

For a heavy object, if $Mc^2 > k_B T_{\text{brane}}$, the infinite thermalization condition cannot be satisfied. Here c is light-velocity on 3-brane. We cannot use the formula of a fragmentized knot under waving brane-twisting to describe the heavy object. So, for an object, there exists a crossover from quantum mechanics to classical mechanics. The crossover can be estimated by the mass

$$M_{\text{crossover}} c^2 \sim k_B T_{\text{brane}}. \quad (145)$$

In the limit of $Mc^2 \ll k_B T_{\text{brane}}$, the object does quantum motions; in the limit of $Mc^2 \gg k_B T_{\text{brane}}$, the object does classical motions, i.e.,

$$\text{Knot physics} \implies \begin{cases} \text{classical mechanics, } Mc^2 \gg k_B T_{\text{brane}} \\ \text{quantum, mechanics, } Mc^2 \ll k_B T_{\text{brane}} \end{cases}. \quad (146)$$

In the limit of $Mc^2 \gg k_B T_{\text{brane}}$, the energy comes from shifting motion may be larger than that from brane-twisting, i.e., $E_{\text{shift}} = \frac{1}{2} M (\vec{v}_{\text{shift}})^2 \gg E_{\text{twist}} = \hbar\omega$. As a result, $E_{\text{total}} \simeq E_{\text{shift}} \neq E_{\text{twist}}$, the heavy object doesn't obey the Schrödinger equation. Instead, it obeys classical mechanics.

We point out that knot mechanics is a *nonlinear mechanics* that is different from classical mechanics and quantum mechanics that are all linear mechanics. For a linear mechanics, a composite object obeys the same mechanics to that of its constituent – A composite classical object obeys classical mechanics and a composite quantum object obeys quantum mechanics. However, for knot mechanics, single knot obeys quantum mechanics but a composite object with many-knot may obey classical mechanics. This abnormal property for knots is due to their contradictoriness character.

From above discussion, we solve *the Schrödinger's cat paradox*. A cat is a classical object that doesn't obey quantum mechanics. So there is no Schrödinger's cat paradox at all.

2. Application – measurement theory in knot mechanics

Measurement is an important issue of quantum mechanics. P.A.M. Dirac (1958) in "*The Principles of Quantum Mechanics said, "A measurement always causes the system to jump into an eigenstate of the dynamical variable that*

is being measured, the eigenvalue this eigenstate belongs to being equal to the result of the measurement.” According to the Copenhagen interpretation, the state of a system is assumed to “collapse” into an eigenstate of the operator during measurement. That is the so-called measurement process of “wave-function collapse”. The wave-function collapse is random and indeterministic and the predicted value of the measurement is described by a probability distribution. The wave-function collapse raises “the measurement problem”, as well as questions of determinism and locality, as demonstrated in the Einstein–Podolsky–Rosen (EPR) paradox[18].

In this section, we study the measurement problem by using knot mechanics. In knot mechanics, the aim of measurement is to measure the pattern of the waving brane-twisting motion that is described by the wave-function

$$\frac{[z_B(\phi_B)]_{\text{fragment}}}{\sqrt{V}|z_B|} \Rightarrow \psi(\vec{x}, t) = \sqrt{\Omega(\vec{x}, t)} e^{i\phi(\vec{x}, t)}. \quad (147)$$

In physics, such wave-function $\psi(\vec{x}, t)$ denotes *smooth, slow, non-topological* phase changing from waving brane-twisting. To determine it, we try to capture the fragmentized (or randomized) knots and have their probability distribution. Due to complementarity principle, a knot is π -phase changing with a phase angle $\Delta\phi(\vec{x}, t)$.

Measurement actually determines knots by projecting operation. Because a knot is really a sharp phase changing that could interact with the measuring devices. Due to infinite thermalization condition, a knot is fragmentized (or randomized), of which the knot-function is defined by

$$[z_B(\phi_B)]_{\text{fragment}, N} = \prod_{i=1}^N \hat{U}(\Delta\phi_B = \frac{\pi}{N}, (x_0)_i) z_B. \quad (148)$$

To find a fragmentized (or randomized) knot, we need to do full-projection

$$\begin{aligned} \hat{P}_F[z_A(x_1, x_2, \dots, x_d)] \\ = \hat{P}_F[z_B(\phi_B(x_1, x_2, \dots, x_d))_{\text{fragment}, N}]. \end{aligned} \quad (149)$$

Under a full-projection, a fragmentized (or randomized) knot is fixed at x_0 . During the full-projection, we obtain the exact position of a knot x_0 and the fragmentized knot $z_B(\phi_{0,B}(x_0))_{\text{fragment}, N}$ turns into a full knot $z_B(\phi_{0,B}(x_0))$, i.e.,

$$\begin{aligned} z_B(\phi_{0,B}(x_0))_{\text{fragment}, N} &= \prod_{i=1}^N \hat{U}(\Delta\phi_B = \frac{\pi}{N}, (x_0)_i) z_B \\ &\rightarrow \hat{U}(\phi_0, \Delta\phi_B = \frac{\pi}{N}, (x_0)_i) z_B \\ &\rightarrow z_B(\phi_{0,B}(x_0)). \end{aligned} \quad (150)$$

In addition, the quantum phase of a (projected) knot is fixed at ϕ_0 . Therefore, the full-projection process corresponds to “wave-function collapse”. When the measurement occurs, the position of a knot and the phase angle of quantum state are all fixed. When the position $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_d)$ and the phase angle are fixed, a knot cannot be regarded as π -phase changing anymore ($\phi(\vec{x}, t) = \phi_0$, or $\Delta\phi(\vec{x}, t) \rightarrow 0$). Instead, *a knot becomes a classical topological configuration after measuring*. This gives the explanation of “wave-function collapse” by using knot mechanics. Thus, the measure process in knot mechanics is summarized by the following equation

$$\begin{aligned} &\text{Projecting a fragmentized knot} \\ &\Rightarrow \text{Finding position and phase of a knot} \\ &\Rightarrow \text{A knot becomes a classical} \\ &\text{topological configuration.} \end{aligned} \quad (151)$$

In addition, we discuss another fundamental principle in quantum mechanics – *uncertainty principle*. according to matter-motion complementarity principle, a knot is topological phase changing ($\Delta\phi = \pi$) with fixed phase angle ($\phi = \phi_0$). So, in knot mechanics, we cannot exactly determine the phase angle of a knot by observing phase changing. We call this property to be *matter-motion complementarity principle* that leads to *uncertainty principle* in quantum mechanics.

We may give an explanation on *the double-slit experiment*. Before measurement in double-slit experiment, the knot can be regarded as fragmentized knot, of which the probability distribution is described by the wave-function. It has no classical path. See the illustration of double-slit experiment in Fig.8.(a), in which there exists particular interference pattern on the screen that agrees to the prediction from quantum mechanics. However, after measurement, we know the knot moving through a slit. The knot cannot be regarded as a fragmentized knot but a classical one. As a result, the interference disappears. Fig.8.(b) is an illustration of detective double-slit experiment, in which there doesn’t exist interference pattern on the screen.

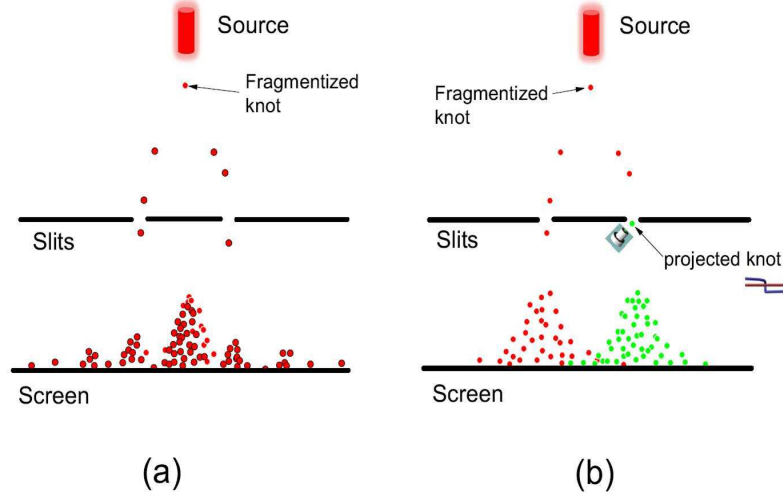


FIG. 8: (a) An illustration of free double-slit experiment. There exists particular interference pattern on the screen that agrees to the prediction from quantum mechanics; (b) An illustration of detective double-slit experiment. There doesn't exist interference pattern on the screen. The red spots denote fragmentized knots (knots obey quantum mechanics) and the green spots denote projected knot (knots obey classical mechanics).

G. Conclusion

In the end of this section, we draw the conclusion. We point out that our universe is two entangled 3-branes in 5D $\{x, y, z, \xi, \eta\}$ space. The elementary particles (for example electrons) are knots, half-windings between two 3-branes with phase angle. According to topological nature of a knot, it is stable against local perturbations and could not be divided into smaller parts. There are several degrees of freedom for a knot: the phase angle, the position and winding direction in extra dimensional space (phase ϕ and \vec{e} , respectively) on 3-branes. There are four key properties of knots in knot mechanics:

1. **Phase changing:** A knot is a half-winding between two d -branes in $d + 2$ dimensional space. Or, a knot (an elementary particle) is π phase-changing along a given direction \vec{e} in d dimensional space;
2. **Phase angle:** A knot has a phase angle, ϕ_0 ;
3. **Topological configuration of a classical complex field:** A static knot is a topological configuration of a classical complex field. This property is one of key point to the understanding of spin degrees of freedom and fermionic statistics of a knot;
4. **Motions:** There are two types of motions of a knot: knot-shifting and brane-twisting. Brane-twisting is non-local motion. Under infinite thermalization condition, the knot mechanics for dynamics of waving brane-twisting is just quantum mechanics.

There are two fundamental principles in knot mechanics – **matter-motion complementarity principle** and **quantum-classical contradictoriness**. Matter-motion complementarity principle is a principle to describe the relationship between matter (knots) and its motion. Based on matter-motion complementarity principle, matter (knot – π -phase changing) and its motion (smooth phase changing) are unified into a phenomenon – entanglements between d -branes. Quantum-classical contradictoriness emphasizes that knot mechanics is a nonlinear mechanics that unifies quantum mechanics and classical mechanics under infinite thermalization condition. Therefore, the knot is "chameleon-like" matter. We say the answer of "what is the matter" is determined by the answer of "how matter moves".

III. KNOT-CRYSTAL THEORY

Field is an important concept in physics. Quantum field theory, particular the quantum electrodynamics (QED) is an extremely successful theory of electromagnetic interaction that agrees with experiments very well. The *Standard model* (SM) of $SU_{\text{Strong}}(3) \otimes SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge theory is a special type of quantum field theory that focuses on three non-gravitational interactions: weak interaction, strong interaction, and electromagnetic interaction[1]. The SM beyond QED is also verified to high precision.

Although SM is a successful theory, people study the underlying physics of quantum field theory. However, because SM is a complicated gauge theory with up to 19 free parameters, people try to develop a theory beyond SM. *Grand unified theory* (GUTs) is proposed to unite SM physics through a quantum gauge theory with a larger gauge symmetry at higher energy (for example, $SU(5)$ gauge theory[19]).

In this part, we will answer question 3 and question 4 within a unified picture and point out a new approach towards understanding quantum field theory and SM. We call the new theory *knot-crystal theory*. From this point of view, to find a microscopic origin of SM is to find a particular organization of knots – *knot-crystal*, such that the collective motions of knot-crystal are described by fermionic elementary particles and gauge fields. Fermionic elementary particles are topological excitations that correspond to different types of *composite knots* (knots with *internal-windings*). The collective quantum fluctuations of the internal-windings become gauge fields – position-fluctuations of internal-windings are Abelian gauge field and order-fluctuations of internal-windings are non-Abelian gauge field. Two composite knots interact by exchanging quantum fluctuations of the internal-windings. The generalized translation symmetry of composite knot-crystal guarantees local (Abelian/nonAbelian) gauge symmetries for the composite knots.

This part is organized as below. In Sec. A, we review knot mechanics. In Sec. B, we give the mathematical definition of knot-crystal – periodic entanglements between two d -branes and introduce the classification of knot-crystals. In Sec. C, we study 1-level knot-crystal and found that for a particular knot-crystal – 1-level double-helix knot-crystal, the effective theory is Dirac model and the collective motions of knots are described by Dirac equation. In Sec. D, we point out that the theory for the composite knot-crystal is an interacting model of Dirac field coupling $SU_{\text{Strong}}(n) \otimes U_{\text{em}}(1)$ gauge fields and the knots in 1-level composite knot-crystal become elementary particles including neutrino, electron and quarks. The gauge fields can be viewed as fluctuations of the internal-windings. In Sec. E, based on 3-level double-helix knot-crystal, we discuss $SU_{\text{weak}}(2) \otimes U_Y(1)$ weak interaction and Higgs mechanism. In Sec. F, we show that our universe is a standard knot-crystal, of which the low energy effective theory is the Standard model – an $SU_{\text{Strong}}(3) \otimes SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge theory with Higgs mechanism due to spontaneous symmetry breaking. Finally, the conclusions are drawn in Sec. G.

A. Review on knot mechanics

It was known that the elementary particles in physics (for example electrons) are knots, local half-windings of one brane around another along an arbitrary direction of brane in $d + 2$ D space. When looking it far away, people find a tiny ball; When approaching it, people find the detailed structure – an entanglement between two d -branes in $d + 2$ dimensional ($d + 2$ D) space. According to the correspondence between knot-functions $z_i(x) = r_i(x) \exp(i\phi_i(x))$ and classical complex field on a flat d -brane $\Phi_i(x)$, i.e.,

$$z_i(x) \implies \Phi_i(x), \quad (152)$$

knots correspond to topological configurations of classical complex field in d dimensional space: 1D knot to a π -domain wall; 2D knot to a 2π -flux; 3D knot to a Dirac monopole.

Before discussing knot-crystal, we summarize knot's properties. A knot is an entanglement between two d -branes. On the one hand, a knot is π -phase changing – a *sharp, time-independent, topological* phase changing, on the other hand, a knot has a phase angle. Under infinite thermalization condition, the knot becomes fragmentized knot and obeys quantum mechanics. Quantum mechanics describes the dynamics of *smooth, slow, non-topological* phase changing of knots. For a waving brane-twisting of a given pattern (ω, \vec{k}) , the knot-function

$$\frac{[z_B(\phi_B)]_{\text{fragment}}}{\sqrt{V}|z_B|} \rightarrow \psi(\vec{x}, t) = \sqrt{\Omega(\vec{x}, t)} e^{i\phi(\vec{x}, t)} \quad (153)$$

plays the role of wave-function in quantum mechanics where $\Omega(\vec{x}, t)$ is the particle probability and $\phi(\vec{x}, t)$ is the quantum phase. The energy and momentum for waving brane-twisting are described by operators

$$E \rightarrow \hat{E} = -i \frac{d}{dt}, \quad \vec{p} \rightarrow \hat{p} = i \frac{d}{d\vec{x}}. \quad (154)$$

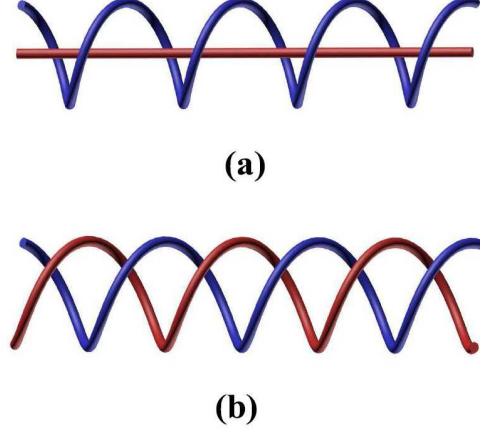


FIG. 9: (a) A 1D chiral knot-crystal; (b) A 1D double-helix knot-crystal.

The energy-momentum relationship $E = H(\vec{p})$ becomes the equation of motion for wave-function (twisting motion),

$$\hat{E}\psi(\vec{x}) = \hat{H}(\vec{p})\psi(\vec{x}). \quad (155)$$

There exists Planck's relationship $E = \hbar\omega$ where \hbar takes place of the random strength such as temperature for thermalize brane. The new Planck "constant" will be determined by the system, such as the temperature of a system and the size of winding region.

We may use path-integral formulation to describe quantum processes in knot mechanics. For a multi-knot system, the probability amplitude is defined by

$$\begin{aligned} & \langle t_f, \vec{x}'_M, \dots, \vec{x}'_2, \vec{x}'_1 | t_i, \vec{x}_M, \dots, \vec{x}_2, \vec{x}_1 \rangle \\ &= \int \mathcal{D}\psi^\dagger(\vec{x}, t) \mathcal{D}\psi(\vec{x}, t) e^{i\mathcal{S}/\hbar} \end{aligned} \quad (156)$$

where $\mathcal{S} = \sum_{\omega, \vec{p}} S_{\omega, \vec{p}} = \int \mathcal{L} dt d^3x$ with $S_{\omega, \vec{p}} = \psi^\dagger_{\vec{p}}(i\hbar\omega(\vec{p}) - H(\vec{p}))\psi_{\vec{p}}$ and $\mathcal{L} = i\psi^\dagger \partial_t \psi - \hat{\mathcal{H}}$.

B. Knot-crystal – periodic entanglement-pattern between d -branes

In solid state physics, a basic theory is about atom-crystal and its lattices. The atom-crystal has its inherent symmetry, by which we may classify different types of symmetries (translational symmetry, rotation symmetry, mirror symmetry). For example, there are 230 distinct space groups in 3D space. In addition to simple crystal with monatomic lattices, there exist composite crystals with polyatomic lattices that have more than one type of atoms in a unit cell.

In this paper, a periodic entanglement-pattern between two d -branes is called *knot-crystal*. The definition of a "knot-crystal" is based on periodic structures of knots that is similar to atom-crystal where the atoms form a periodic arrangement. However, knot-crystal is different from traditional atom-crystal. Fig.9 shows two types of 1D knot-crystals in 3D space. Due to the existence of rotation symmetry and generalized translation symmetry, the properties of knot-crystals are much different from that of atom-crystals.

1. Definition

Firstly we consider an arbitrary d dimensional multi-knot system that comes from a periodic entanglement pattern by winding d -branes in $d + 2$ dimensional space $(x_1, x_2, \dots, x_d, x_{d+1}, x_{d+2})$. To characterize a knot-crystal, we define

the knot-function by an $\mathcal{N} \times \mathcal{M}$ matrix

$$\mathbf{Z}(\vec{e}, \vec{x}, t) = \begin{pmatrix} \mathbf{z}_1(\vec{e}, \vec{x}, t) \\ \mathbf{z}_2(\vec{e}, \vec{x}, t) \\ \dots \\ \mathbf{z}_{\mathcal{N}}(\vec{e}, \vec{x}, t) \end{pmatrix} = \begin{pmatrix} z_{1,1}(\vec{x}, t) & z_{1,2}(\vec{x}, t) & \dots & z_{1,\mathcal{M}}(\vec{x}, t) \\ z_{2,1}(\vec{x}, t) & z_{2,2}(\vec{x}, t) & \dots & \dots \\ \dots & \dots & \dots & \dots \\ z_{\mathcal{N},1}(\vec{x}, t) & z_{\mathcal{N},2}(\vec{x}, t) & \dots & z_{\mathcal{N},\mathcal{M}}(\vec{x}, t) \end{pmatrix} \quad (157)$$

where $\vec{x} = (x_1, x_2, \dots, x_d)$ and \vec{e} is the winding direction. \mathcal{N} and \mathcal{M} denote the brane-index and the level-index, respectively. We point out that $\mathbf{Z}(\vec{e}, \vec{x}, t)$ is just matrix representation to show the properties of knot-crystal and there may exist $\mathcal{N} \times \mathcal{M}$ independent knot-functions. $\mathbf{z}_j(\vec{e}, \vec{x}, t)$ denotes the knot-function of j -th brane.

A generalized definition of a knot-crystal is given by the knot-function of j -th level i -th brane as

$$z_{i,j}(\vec{e}, \vec{x}, t) = r_{i,j} e^{i\Delta\phi_{i,j} + i\omega_{i,j}t + i(\phi_{i,j})_0} \quad (158)$$

where $\Delta\phi_{i,j}$ and $(\phi_{i,j})_0$ denote the spacial-dependent phase angle and the constant phase angle, respectively. $\omega_{i,j}$ is rotating velocity and $r_{i,j}$ is the radius of j -th level winding of i -th brane, respectively. In general, we have $r_{i,j} \simeq r_{i',j}$ ($j = 1, i \neq i'$) and $r_{i,j} \equiv r_{i',j}$ ($j \neq 1, i \neq i'$).

To determine a knot-crystal, there is a *hierarchy recurrence relationship*

$$\begin{aligned} \Delta\phi_{i,\mathcal{M}} &= \pm \vec{k}_{i,0} \cdot \vec{x}, \\ \Delta\phi_{i,\mathcal{M}-1} &= \pm \Delta_{i,(\mathcal{M}-1,\mathcal{M})} \Delta\phi_{i,\mathcal{M}}, \\ &\dots \\ \Delta\phi_{i,j-1} &= \pm \Delta_{i,(j-1,j)} \Delta\phi_{i,j}, \\ &\dots \\ \Delta\phi_{i,1} &= \pm \Delta_{i,(1,2)} \Delta\phi_{i,2}, \end{aligned} \quad (159)$$

where $\vec{k}_{i,0} = \pm \frac{\pi}{a_{i,\mathcal{M}}} \vec{e}$ denotes winding vector along the winding direction, and $a_{i,\mathcal{M}}$ is the length that denotes the half pitch of largest windings of i -th brane. \pm denotes the winding chirality. $\Delta_{i,(j-1,j)}$ is the positive winding number of $(j-1)$ -th level windings in a j -th level winding for i -th brane.

Thus, according to the hierarchy recurrence relationship, we have

$$z_{i,j}(\vec{e}, \vec{x}, t) = r_{i,j} \cdot e^{\pm i[\prod_j^{\mathcal{M}} \Delta_{i,(j-1,j)}] \cdot \phi_{i,\mathcal{M}}} \cdot e^{i\omega_{i,j}t} \cdot e^{i(\phi_{i,j})_0}. \quad (160)$$

As a result, the knot-function for a knot-crystal with \mathcal{M} -level \mathcal{N} branes is given by

$$\mathbf{Z}(\vec{e}, \vec{x}, t) = \begin{pmatrix} r_{1,1} e^{i\Delta\phi_{1,1} + i\omega_{1,1}t + i(\phi_{1,1})_0} & r_{12} e^{i\Delta\phi_{1,2} + i\omega_{1,2}t + i(\phi_{1,2})_0} & \dots & r_{1,\mathcal{M}} e^{i\Delta\phi_{1,\mathcal{M}} + i\omega_{1,\mathcal{M}}t + i(\phi_{1,\mathcal{M})_0} \\ r_{2,1} e^{i\Delta\phi_{2,1} + i\omega_{2,1}t + i(\phi_{2,1})_0} & r_{22} e^{i\Delta\phi_{2,2} + i\omega_{2,2}t + i(\phi_{2,2})_0} & \dots & r_{2,\mathcal{M}} e^{i\Delta\phi_{2,\mathcal{M}} + i\omega_{2,\mathcal{M}}t + i(\phi_{2,\mathcal{M})_0} \\ \dots & \dots & \dots & \dots \\ r_{\mathcal{N},1} e^{i\Delta\phi_{\mathcal{N},1} + i\omega_{\mathcal{N},1}t + i(\phi_{\mathcal{N},1})_0} & r_{\mathcal{N},2} e^{i\Delta\phi_{\mathcal{N},2} + i\omega_{\mathcal{N},2}t + i(\phi_{\mathcal{N},2})_0} & \dots & r_{\mathcal{N},\mathcal{M}} e^{i\Delta\phi_{\mathcal{N},\mathcal{M}} + i\omega_{\mathcal{N},\mathcal{M}}t + i(\phi_{\mathcal{N},\mathcal{M})_0} \end{pmatrix} \quad (161)$$

where

$$\Delta\phi_{i,j}(\phi_{i,\mathcal{M}}, t) = \pm [\prod_j^{\mathcal{M}} \Delta_{i,(j-1,j)}] \cdot \phi_{i,\mathcal{M}}. \quad (162)$$

Due to spacial rotation symmetry, we may choose different winding directions by doing spacial rotation transformation \hat{R}_{space} as $\vec{e}' = \hat{R}_{\text{space}} \cdot \vec{e}$ or $\vec{k}'_0 = \hat{R}_{\text{space}} \cdot \vec{k}_0$. The knot physics is invariant under spacial rotation transformation as

$$\begin{aligned} \mathbf{Z}(\vec{e}, \vec{x}, t) &\rightarrow \mathbf{Z}'(\vec{e}', \vec{x}', t) = \hat{R}_{\text{space}} \cdot \mathbf{Z}(\vec{e}, \vec{x}, t) \\ &= \mathbf{Z}(\vec{e}', \vec{x}', t). \end{aligned} \quad (163)$$

2. Classification

To classify a knot-crystal, we introduce four indices: *dimension*, *brane number*, *chirality*, *level*.

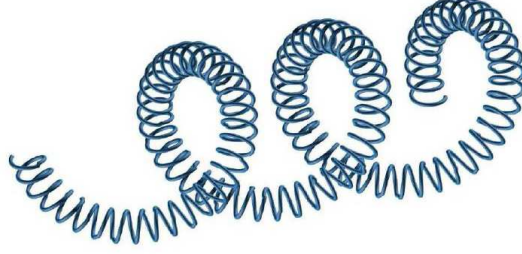


FIG. 10: An illustration of a 2-level knot-crystal.

The first index is dimension of knot-crystal, d . In this paper we focus on 3D knot-crystal ($d = 3$) that is described by $\mathbf{Z}(\vec{e}, \vec{x}, t)$ where $\vec{x} = (x, y, z)$. Fig.9 is an illustration of two types of 1D knot-crystals in 3D space.

The second index is the number of d-branes, \mathcal{N} . Because each d-brane is mapped onto a complex field, a knot-crystal with \mathcal{N} d-branes can be mapped onto \mathcal{N} -component complex field. For a knot-crystal with $\mathcal{N} = 1$, a knot-function is

$$\begin{aligned} \mathbf{Z}(\vec{e}, \vec{x}, t) &= \begin{pmatrix} 0 \\ \mathbf{z}(\vec{e}, \vec{x}, t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \dots & 0 \\ r_1 e^{i\Delta\phi_1 + i\omega_1 t + i(\phi_1)_0} & r_2 e^{i\Delta\phi_2 + i\omega_2 t + i(\phi_2)_0} & \dots & r_{\mathcal{M}} e^{i\Delta\phi_{\mathcal{M}} + i\omega_{\mathcal{M}} t + i(\phi_{\mathcal{M}})_0} \end{pmatrix}. \end{aligned} \quad (164)$$

In this paper we only consider knot-crystals with *two* 3D 3-branes ($\mathcal{N} = 2$), of which a generalized knot-function is given by

$$\begin{aligned} \mathbf{Z}(\vec{e}, \vec{x}, t) &= \begin{pmatrix} \mathbf{z}_A(\vec{e}, \vec{x}, t) \\ \mathbf{z}_B(\vec{e}, \vec{x}, t) \end{pmatrix} \\ &= \begin{pmatrix} r_{A,1} e^{i\Delta\phi_{A,1} + i\omega_{A,1} t + i(\phi_{A,1})_0} & r_{A,2} e^{i\Delta\phi_{A,2} + i\omega_{A,2} t + i(\phi_{A,2})_0} & \dots & r_{A,\mathcal{M}} e^{\pm i\vec{k}_{(A,\mathcal{M}),0} \cdot \vec{x} + i\omega_{A,\mathcal{M}} t + i(\phi_{A,\mathcal{M}})_0} \\ r_{B,1} e^{i\Delta\phi_{B,1} + i\omega_{B,1} t + i(\phi_{B,1})_0} & r_{B,2} e^{i\Delta\phi_{B,2} + i\omega_{B,2} t + i(\phi_{B,2})_0} & \dots & r_{B,\mathcal{M}} e^{\pm i\vec{k}_{(B,\mathcal{M}),0} \cdot \vec{x} + i\omega_{B,\mathcal{M}} t + i(\phi_{B,\mathcal{M}})_0} \end{pmatrix}. \end{aligned} \quad (165)$$

We call a 2-level knot-crystal to be *double-helix knot-crystal*. Fig.9 is an illustration of two types of 1D knot-crystals.

The third index is *level* of knot-crystal, a very important feature of knot-crystal. To define the concept of level, we introduce *composite* knot-crystal that corresponds to polyatomic atom-crystal with a composite lattice. Fig.10 is an illustration of a 2-level knot-crystal with $\mathcal{N} = 1$. Thus, for an \mathcal{M} -level 3D double-helix knot-crystal, we can label the knot-crystals by the following hierarchy recurrence relationship,

$$\begin{aligned} \Delta\phi_{i,\mathcal{M}} &\implies \pm \frac{\pi}{a_{i,\mathcal{M}}} \vec{e} \cdot \vec{x} \\ &\implies \Delta\phi_{i,\mathcal{M}-1} = \pm \Delta_{i,(\mathcal{M}-1,\mathcal{M})} \Delta\phi_{i,\mathcal{M}} \\ &\implies \dots \\ &\implies \Delta\phi_{i,2} = \pm \Delta_{i,(2,3)} \Delta\phi_{i,3} \\ &\implies \Delta\phi_{i,1} = \pm \Delta_{i,(1,2)} \Delta\phi_{i,2}. \end{aligned} \quad (166)$$

We simplify the hierarchy recurrence relationship by two series of numbers arranged according to a given hierarchy series

$$\begin{pmatrix} \{\Delta_{A,(1,2)}, \Delta_{A,(2,3)}, \dots, \Delta_{A,(\mathcal{M}-1,\mathcal{M})}\} \\ \{\Delta_{B,(1,2)}, \Delta_{B,(2,3)}, \dots, \Delta_{B,(\mathcal{M}-1,\mathcal{M})}\} \end{pmatrix}. \quad (167)$$

The fourth index is chirality of branes in knot-crystals. For i -th brane of a \mathcal{M} -level knot-crystal, the hierarchy recurrence relationship is given by

$$\{\Delta_{i,(1,2)}, \Delta_{i,(2,3)}, \dots, \Delta_{i,(\mathcal{M}-1,\mathcal{M})}\}. \quad (168)$$

There exist $2^{\mathcal{M}}$ possible chiralities for different signs \pm as

$$\mathbf{z}_{i\text{-th}}(\vec{e}, \vec{x}, t) = \left(r_{i,1} e^{i\Delta\phi_{i,1} + i\omega_{i,1}t + i(\phi_{i,1})_0} \quad r_{i,2} e^{i\Delta\phi_{i,2} + i\omega_{i,2}t + i(\phi_{i,2})_0} \quad \dots \quad r_{i,\mathcal{M}} e^{i\Delta\phi_{i,\mathcal{M}} + i\omega_{i,\mathcal{M}}t + i(\phi_{i,\mathcal{M}})_0} \right) \quad (169)$$

with

$$\begin{aligned} \Delta\phi_{i,\mathcal{M}} &= \pm \frac{\pi}{a_{i,\mathcal{M}}} \vec{e}, \dots \\ \Delta\phi_{i,2} &= \pm \Delta_{i,(2,3)} \Delta\phi_{i,3} \\ \Delta\phi_{i,1} &= \pm \Delta_{i,(1,2)} \Delta\phi_{i,2}. \end{aligned} \quad (170)$$

There are two possible chiralities for a 1-level knot-crystal with $\mathcal{N} = 1$: one is called left-hand knot-crystal with clockwise winding, of which the knot-function is

$$\mathbf{Z}_L(\vec{e}, \vec{x}, t) = \begin{pmatrix} 0 \\ \mathbf{z}_B(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} 0 \\ r_0 e^{i\vec{k}_0 \cdot \vec{x} + i\omega t + i\phi_0} \end{pmatrix}, \quad (171)$$

the other right-hand knot-crystal with anti-clockwise winding, of which the knot-function is

$$\mathbf{Z}_R(\vec{e}, \vec{x}, t) = \begin{pmatrix} 0 \\ \mathbf{z}_B(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} 0 \\ r_0 e^{-i\vec{k}_0 \cdot \vec{x} + i\omega t + i\phi_0} \end{pmatrix}. \quad (172)$$

For double-helix knot-crystal with $\mathcal{M} = 1$, there are four possible chiralities

$$\mathbf{Z}(\vec{e}, \vec{x}, t) = \begin{pmatrix} \mathbf{z}_A(\vec{x}, t) \\ \mathbf{z}_B(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} r_A e^{\pm i\vec{k}_0 \cdot \vec{x} + i\omega_A t + i\phi_{A,0}} \\ r_B e^{\pm i\vec{k}_0 \cdot \vec{x} + i\omega_B t + i\phi_{B,0}} \end{pmatrix} \quad (173)$$

where $\phi_{A,0}$ and $\phi_{B,0}$ are constant. See the illustration of helix knot-crystal in Fig.9.(b).

In particular, our universe is 3D 3-level double-helix knot-crystal ($\mathcal{M} = 3$), of which the hierarchy series is

$$\begin{pmatrix} \{3.5, N\} \\ \{3.5, N + \delta\} \end{pmatrix} \quad (174)$$

where N is a very large number, $N \gg 1$ (for example, $N \sim 10^{15}$) and δ is a very tiny number, $\delta \ll 1$ (for example, $\delta \sim 10^{-10}$). For this reason, the hierarchy series is really a code of our universe. We call it *hierarchy series of universe*.

3. Knot-crystal-operation and the superposition principle for knot-crystal

In this paper, to characterize the superposition principle of knot-crystals, we introduce the knot-crystal-operation. Firstly, we generate \mathcal{M} -th level knot-crystal from \mathcal{N} -flat branes, of which the knot-crystal-operation is defined by

$$\hat{U}(z_{i,\mathcal{M}}(\vec{x}, t)) = \exp[i[\Delta\phi_{i,\mathcal{M}}(\vec{x}, t) + \omega_{i,\mathcal{M}}t + (\phi_{i,\mathcal{M}})_0] \cdot \hat{K}] \quad (175)$$

where $\hat{K} = -i \frac{d}{d\phi_{i,\mathcal{M}}}$ and

$$\Delta\phi_{i,\mathcal{M}}(\vec{x}, t) = \pm \vec{k}_{i,0} \cdot \vec{x}. \quad (176)$$

$\vec{k}_{i,0} = \frac{\pi}{a_{i,\mathcal{M}}} \vec{e}$ denotes winding vector along the winding direction, and $a_{i,\mathcal{M}}$ is the length that denotes the half pitch of largest windings of i -th brane. \pm denotes the winding chirality. \vec{e} is a given direction and x is the coordination on the axis along a given direction \vec{e} (or $x \cdot \vec{e} = \vec{x}$), $\omega_{i,\mathcal{M}}$ is rotating velocity, $(\phi_{i,\mathcal{M}})_0$ is a constant phase angle of \mathcal{M} -th level i -th brane of a knot-crystal. As a result, the knot-function of \mathcal{M} -th level knot-crystal becomes

$$z_{i,\mathcal{M}}(\vec{x}, t) = \hat{U}(z_{i,\mathcal{M}}(\vec{x}, t))(r_{i,\mathcal{M}}) = (r_{i,\mathcal{M}} \exp[i[\Delta\phi_{i,\mathcal{M}}(\vec{x}, t) + \omega_{i,\mathcal{M}}t + (\phi_{i,\mathcal{M}})_0]]). \quad (177)$$

Second, based on \mathcal{M} -th level knot-crystal, we generate $(\mathcal{M} - 1)$ -th level knot-crystal, of which the knot-crystal-operation is defined by

$$\hat{U}(z_{i,\mathcal{M}-1}(\vec{x}, t)) = \exp[i[\Delta\phi_{i,\mathcal{M}-1}(\vec{x}, t) + \omega_{j,\mathcal{M}-1}t + (\phi_{j,\mathcal{M}-1})_0] \cdot \hat{K}] \quad (178)$$

where $\hat{K} = -i\frac{d}{d\phi_{i,\mathcal{M}-1}}$ and

$$\Delta\phi_{i,\mathcal{M}-1}(\Delta\phi_{i,\mathcal{M}}, t) = \Delta_{i,(\mathcal{M}-1,\mathcal{M})} \cdot \Delta\phi_{i,\mathcal{M}}. \quad (179)$$

As a result, the knot-function of $(\mathcal{M} - 1)$ -th level knot-crystal becomes

$$\begin{aligned} z_{i,\mathcal{M}-1}(\vec{x}, t) &= \hat{U}(z_{i,\mathcal{M}-1}(\vec{x}, t))(r_{i,\mathcal{M}-1}) \\ &= (r_{i,\mathcal{M}-1}) \exp[i[\Delta\phi_{i,\mathcal{M}}(r_{i,\mathcal{M}}, t) + \omega_{i,\mathcal{M}}t + (\phi_{i,\mathcal{M}})_0]]. \end{aligned} \quad (180)$$

By using this approach, we may generate the full knot-crystal on \mathcal{N} -flat branes level-by-level as

$$\mathbf{Z}(\vec{e}, \vec{x}, t) = \hat{U}(\mathbf{Z}(\vec{e}, \vec{x}, t)) \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} & \dots & r_{1,\mathcal{M}} \\ r_{2,1} & r_{2,2} & r_{2,3} & \dots & r_{2,\mathcal{M}} \\ \dots & \dots & \dots & \dots & \dots \\ r_{\mathcal{N},1} & r_{\mathcal{N},2} & r_{\mathcal{N},3} & \dots & r_{\mathcal{N},\mathcal{M}} \end{pmatrix} \quad (181)$$

where knot-crystal-operation $\hat{U}(\mathbf{Z}(\vec{x}, t))$ is defined by

$$\hat{U}(\mathbf{Z}(\vec{x}, t)) = \hat{L}[\prod_{i,j} \hat{U}(z_{i,j}(\vec{x}, t))] \quad (182)$$

where \hat{L} is an operator to keep the order of knot-crystal-operation from higher level to lower level.

Due to the Abelian character of the knot-crystal-operation $\hat{U}(\mathbf{z}_{i,j}(\vec{x}, t))$, there exist the following commutation relations for knot-crystal-operations of same level (for example, i -th level), i.e.,

$$[\hat{U}(\mathbf{z}_{i,j}(\vec{x}, t)), \hat{U}(\mathbf{z}_{i,l}(\vec{x}, t))] = 0, \quad j \neq l. \quad (183)$$

These commutation relations show the *superposition principle* for knot-crystal. Thus, by fixing the winding direction \vec{e} , we can combine several d dimensional knot-crystals into a single d dimensional knot-crystal or split a single knot-crystal into several knot-crystals.

4. Generalized translation symmetry

One of the most important properties of a knot-crystal is generalized translation symmetry.

It is obvious that all knot-crystals break continuous translation symmetry, i.e.,

$$\mathbf{Z}'(\vec{e}, \vec{x}, t) \neq \mathcal{T}(\Delta\vec{x} \rightarrow 0)\mathbf{Z}(\vec{e}, \vec{x}, t) \quad (184)$$

where $\mathbf{Z}(\vec{e}, \vec{x}, t)$ is knot-function for a 1-level knot-crystal and $\mathcal{T}(\Delta\vec{x})$ is translation operator for knot-functions. All knot-crystals have discrete translation symmetry as

$$\begin{aligned} \mathbf{Z}'(\vec{e}, \vec{x}, t) &= \mathcal{T}(\Delta\vec{x} = 2a)\mathbf{Z}(\vec{e}, \vec{x}, t) \\ &= \mathbf{Z}(\vec{e}, \vec{x} + 2a \cdot \vec{e}, t) = \mathbf{Z}(\vec{e}, \vec{x}, t). \end{aligned} \quad (185)$$

Here a is the length that denotes the half pitch of largest windings of brane. However, we point out that all knot-crystals have *generalized translation symmetry*.

To define the generalized translation symmetry for a knot-crystal, we do a translation operation, under which all branes shift a distance $|\Delta\vec{x}|$ along the direction \vec{e} . i.e.,

$$\begin{aligned} \mathbf{Z}(\vec{e}, \vec{x}, t) &\rightarrow \mathbf{Z}'(\vec{e}, \vec{x}, t) = \mathcal{T}(\Delta\vec{x})\mathbf{Z}(\vec{e}, \vec{x}, t) \\ &= \begin{pmatrix} z_{1,1}(\vec{x}, t)e^{i\Delta\vartheta_{1,1}} & z_{1,2}(\vec{x}, t)e^{i\Delta\vartheta_{1,2}} & \dots & z_{1,\mathcal{M}}(\vec{x}, t)e^{i\Delta\vartheta_{1,\mathcal{M}}} \\ z_{2,1}(\vec{x}, t)e^{i\Delta\vartheta_{2,1}} & z_{2,2}(\vec{x}, t)e^{i\Delta\vartheta_{2,2}} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ z_{\mathcal{N},1}(\vec{x}, t)e^{i\Delta\vartheta_{\mathcal{N},1}} & z_{\mathcal{N},2}(\vec{x}, t)e^{i\Delta\vartheta_{\mathcal{N},2}} & \dots & z_{\mathcal{N},\mathcal{M}}(\vec{x}, t)e^{i\Delta\vartheta_{\mathcal{N},\mathcal{M}}} \end{pmatrix} \end{aligned} \quad (186)$$

where

$$\Delta\vartheta_{\mathcal{N},\mathcal{M}} = \alpha_{\mathcal{N},\mathcal{M}} |\Delta\vec{x}| \quad (187)$$

and $\alpha_{\mathcal{N},\mathcal{M}}$ is a constant for a given brane.

On the other hand, we may show their generalized translation symmetry by considering the knot-crystal-operation on j -th level i -th flat brane of a knot-crystal

$$\begin{aligned} \hat{U}(z_{i,j}(\vec{x}, t)) &= \exp[i[\Delta\phi_{i,j}(\Delta\phi_{i,j+1}, t) \\ &\quad + \omega_{i,j}t + (\phi_{i,j})_0] \cdot \hat{K}] \end{aligned} \quad (188)$$

where $\hat{K} = -i\frac{d}{d\phi_{i,j}}$ and

$$\Delta\phi_{i,j}(\Delta\phi_{i,j+1}, t) = \pm\Delta_{i,(j,j+1)} \cdot \Delta\phi_{i,j+1}(\vec{x}, t), \quad (189)$$

$\omega_{i,j}$ is rotating velocity, $(\phi_{i,j})_0$ is a constant phase angle of j -th level i -th brane of a knot-crystal. To generate a \mathcal{M} -level knot-crystal with \mathcal{N} branes, we do knot-crystal-operation level-by-level. Thus, we do a translation operation $\mathcal{T}(\Delta\vec{x})$ on knot-crystal-operation as

$$\begin{aligned} \hat{U}(z_{i,j}(\vec{x}, t)) &\rightarrow \hat{U}'(z_{i,j}(\vec{x}, t)) \\ &= \mathcal{T}(\Delta\vec{x})\hat{U}(z_{i,j}(\vec{x}, t))\mathcal{T}^{-1}(\Delta\vec{x}) \\ &= \hat{U}(z_{i,j}(\vec{x} + \Delta\vec{x}, t)) = \exp[i\phi_{i,j}(\vec{x} + \Delta\vec{x}, t) \cdot \hat{K}] \\ &= \exp[i\phi_{i,j}(\Delta\vec{x}, t) \cdot \hat{K}] \exp[i\phi_{i,j}(\vec{x}, t) \cdot \hat{K}] \\ &= \exp[\pm i(\frac{\pi}{a_{i,j}}\vec{e} \cdot \Delta\vec{x}) \cdot \hat{K}] \hat{U}(z_{i,j}(\vec{x}, t)) \end{aligned} \quad (190)$$

where $a_{i,j} = \prod_j^{\mathcal{M}} (\Delta_{i,(j-1,j)}) \cdot a_{i,\mathcal{M}}$. As a result, we have

$$\begin{aligned} z_{i,j}(\vec{x}, t) &\rightarrow z'_{i,j}(\vec{x}, t) = \mathcal{T}(\Delta\vec{x})z_{i,j}(\vec{x}, t) \\ &= \mathcal{T}(\Delta\vec{x})\hat{U}(z_{i,j}(\vec{x}, t))(r_{i,j}) \\ &= \mathcal{T}(\Delta\vec{x})\hat{U}(z_{i,j}(\vec{x}, t))\mathcal{T}^{-1}(\Delta\vec{x})\mathcal{T}(\Delta\vec{x})(r_{i,j}) \\ &= \exp[\pm i(\frac{\pi}{a_{i,j}}\vec{e} \cdot \Delta\vec{x})]z_{i,j}(\vec{x}, t). \end{aligned} \quad (191)$$

Because arbitrary knot-crystal could be generated by knot-crystal-operation on flat branes, the translation symmetry of a knot-crystal is determined by the corresponding knot-crystal-operation $\hat{U}(\mathbf{Z}(\vec{e}, \vec{x}, t))$.

C. Knot physics for 1-level double-helix knot-crystal

In this section, we study 1-level knot-crystal. For a 1-level knot-crystal with two branes ($\mathcal{N} = 2, \mathcal{M} = 1$), there exist two different types: one is *chiral knot-crystal*, the other is *double-helix knot-crystal*. Under complex-field- d -Brane correspondence, each knot corresponds to a topological soliton in complex classical field. Thus, knot-crystal corresponds to a *periodic pattern of topological solitons*. We found that for a particular knot-crystal – a 1-level double-helix knot-crystal, the effective theory is Dirac model and the collective motions of knots are described by Dirac equation. For this reason, we also call 1-level double-helix knot-crystal to be *Dirac knot-crystal*.

1. 1-level knot-crystal

a. 1-level chiral knot-crystal Firstly, we define a simplest knot-crystal – a 1-level chiral knot-crystal with $\mathcal{M} = 1$. To show a 1-level chiral knot-crystal, we wind one d -brane around another flat d -brane. The knot-function of a 1-level chiral knot-crystal is

$$\mathbf{Z}_L(\vec{e}, \vec{x}, t) = \begin{pmatrix} z_A(\vec{x}, t) \\ z_{B,L}(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} 0 \\ r_0 e^{i\vec{k}_0 \cdot \vec{x} + i\omega t + i\phi_0} \end{pmatrix} \quad (192)$$

and

$$\mathbf{Z}_R(\vec{e}, \vec{x}, t) = \begin{pmatrix} z_A(\vec{x}, t) \\ z_{B,R}(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} 0 \\ r_0 e^{-i\vec{k}_0 \cdot \vec{x} + i\omega t + i\phi_0} \end{pmatrix} \quad (193)$$

where r_0 is the winding radius of brane-B in extra dimensional space, a is a fixed length that denotes the half pitch of windings and \vec{e} is the winding direction.

Static 1-level chiral knot-crystal We may use knot-crystal-operation to generate a (static) chiral knot-crystal. Firstly, for two types of 1-level knot-crystal, $\mathbf{Z}_L(\vec{e}, \vec{x})$, $\mathbf{Z}_R(\vec{e}, \vec{x})$, we define two knot-crystal-operations $\hat{U}(\mathbf{z}_L)$ and $\hat{U}(\mathbf{z}_R)$ that are

$$\begin{aligned} \hat{U}(\mathbf{z}_L) &= \exp[i\phi_L(\vec{x}) \cdot \hat{K}], \\ \hat{U}(\mathbf{z}_R) &= \exp[i\phi_R(\vec{x}) \cdot \hat{K}], \end{aligned} \quad (194)$$

where $\phi_L(\vec{x}) = \vec{k}_0 \cdot \vec{x} + \phi_0$ and $\phi_R(\vec{x}) = -\vec{k}_0 \cdot \vec{x} + \phi_0$. As a result, $\mathbf{Z}_L(\vec{e}, \vec{x})$ is obtained by

$$\begin{aligned} \mathbf{Z}_L(\vec{e}, \vec{x}) &= \begin{pmatrix} 0 \\ \mathbf{z}_B(\vec{x}) \end{pmatrix} = (0)_A \otimes \mathbf{z}_B(\vec{x}) \\ &= (0)_A \otimes [\hat{U}(\mathbf{z}_L)(1)_B] \end{aligned} \quad (195)$$

and $\mathbf{Z}_R(\vec{e}, \vec{x})$ is obtained by

$$\begin{aligned} \mathbf{Z}_R(\vec{e}, \vec{x}) &= \begin{pmatrix} 0 \\ \mathbf{z}_B(\vec{x}) \end{pmatrix} \\ &= (0)_A \otimes \hat{U}(\mathbf{z}_R)(1)_B \end{aligned} \quad (196)$$

For $\mathbf{Z}_L(\vec{e}, \vec{x})$, we have a left-hand knot-crystal with clockwise winding; For $\mathbf{Z}_R(\vec{e}, \vec{x})$, we have a right-hand knot-crystal with anti-clockwise winding. The two types of knot-crystal can be transformed each other by parity transformation $\hat{\mathcal{P}}$ in extra space,

$$\mathbf{Z}_L(\vec{e}, \vec{x}) = \hat{\mathcal{P}}(x_{d+1}-x_{d+2})\mathbf{Z}_R(\vec{e}, \vec{x}). \quad (197)$$

In particular, in complex representation, the parity transformation in extra space is equal to a complex conjugate transformation

$$\hat{\mathcal{P}}(x_{d+1}-x_{d+2})\mathbf{Z}_R(\vec{e}, \vec{x}) = \mathbf{Z}_R^*(\vec{e}, \vec{x}). \quad (198)$$

According to the knot-equation from out-brane projection

$$\hat{P}_\theta[z_A(\vec{x})] = \hat{P}_\theta[z_B(\vec{x})], \quad (199)$$

we obtain the solutions for a (static) chiral knot-crystal

$$\cos[\phi_B(\vec{x}) - \theta] = 0. \quad (200)$$

We then get

$$\pm \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_0 - \theta = \frac{\pi}{2} + \pi n \quad (201)$$

where n is an arbitrary integer number and θ is projection angle that is fixed to be $-\frac{\pi}{2} + \phi_0$. As a result, the solution for knot is

$$\pm \vec{x}' = a \cdot n + \frac{a}{\pi}(\frac{\pi}{2} - \phi_0 + \theta). \quad (202)$$

By changing the direction of out-brane-projection, we find that the knot-crystal correspondingly shifts. After a period, $\Delta\phi_B(x) = 2\pi$, all knots shift a lattice constant, $2a$.

We take the 1D left-hand static 1-level chiral knot-crystal as an example ($d = 1$). Now, we may consider the 1-brane as a vortex-line in 3D. To generate a 1D knot-crystal, we wind a 1D vortex-line around the other. See the illustration

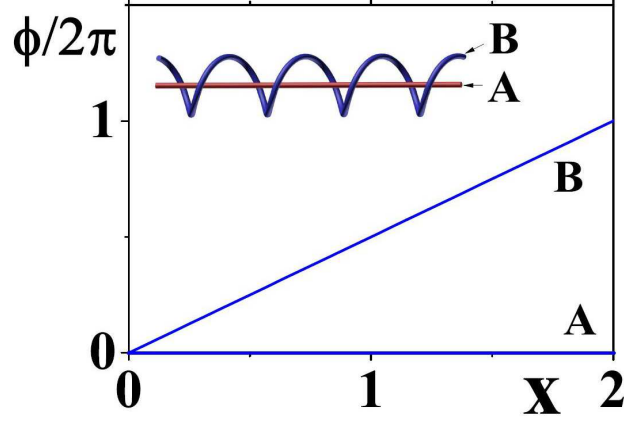


FIG. 11: The phase vs x of a 1D chiral knot-crystal. A and B denote the indices of 1-branes. The scale of x is $\frac{1}{2a}$. The inset is an illustration of 1D chiral knot-crystal, of which the knot function is $\mathbf{Z}(\vec{e}, x) = \begin{pmatrix} z_A(x) \\ z_B(x) \end{pmatrix} = r_0 \begin{pmatrix} 0 \\ e^{ik_0 \cdot x} \end{pmatrix}$.

in Fig.9.(a). In Fig.9.(a), there exist periodic crossings between two vortex-lines. The two 1-branes (a flat 1-brane and a winding 1-brane) are denoted by

$$\mathbf{Z}_L(\vec{e}, \vec{x}) = \begin{pmatrix} z_A(x) \\ z_{B,L}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ r_0 e^{ik_0 \cdot x + i\phi_0} \end{pmatrix} \quad (203)$$

where $z_A(x) = r_0$ and $z_{B,L}(x) = r_0 \exp[i(\phi_0 + \frac{\pi}{a}x)]$, respectively. After projection, $\hat{P}_\theta[z_A(x)] = \hat{P}_\theta[z_{B,L}(x)]$, we obtain

$$\vec{x}' = a \cdot n \quad (204)$$

where n is an integer number and $\theta = -\frac{\pi}{2} + \phi_0$. We may plot the phase vs x in Fig.11. The 1-level chiral knot-crystal is denoted by a line with a slope $\frac{\pi}{a}$.

Rotating 1-level chiral knot-crystal We may also consider a rotating knot-crystal, of which the knot-functions become time-dependent

$$\mathbf{Z}(\vec{e}, \vec{x}, t) = \begin{pmatrix} z_A(\vec{x}, t) \\ z_B(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} 0 \\ r_0 e^{i\phi_B(\vec{x}, t)} \end{pmatrix} \quad (205)$$

where

$$\phi_B(\vec{x}, t) = \pm \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \omega t + \phi_0. \quad (206)$$

From the knot-equation $\cos[\phi_B(x) - \theta] = 0$ and $\theta = -\frac{\pi}{2} + \phi_0$, we have knot-solutions as

$$\pm \vec{x}'(t) = a \cdot n - \frac{a}{\pi} \omega t. \quad (207)$$

Now, for a finite rotating velocity, $\omega \neq 0$, we have a coordinate transformation by a finite velocity of knot-crystal,

$$v = \frac{d\vec{x}'(t)}{dt} = \mp \frac{a}{\pi} \omega. \quad (208)$$

As a result, for a chiral knot-crystal, a knot can *only* be moved by twisting B-brane. This situation corresponds to a particle with fixed chirality (right-hand or left hand). This is the reason that we call this type of knot-crystal to be chiral knot-crystal. So, the knots on chiral knot-crystal will *never* have finite mass. The effective theory of 1-level chiral knot-crystal is described by a quantum field theory of *Weyl fermions*.

Linear knot on chiral knot-crystal A classical linear knot is a half-winding between a flat d -brane (d -brane-A) and a winding d -brane (d -brane-B) along a given direction \vec{e} , of which the knot-function is given by[?]]

$$\mathbf{Z}_{\text{knot}}(\vec{x}, t) = \begin{pmatrix} z_A(\vec{x}, t) \\ z_B(\vec{x}, t) \end{pmatrix}_{\text{knot}} = \begin{pmatrix} 0 \\ r_0 \exp[\pm i\phi_{\text{knot}, B}(x)] \end{pmatrix} \quad (209)$$

where

$$\phi_{\text{knot}, B}(x'_1) = \begin{cases} \phi_0 - \frac{\pi}{2}, & x'_1 \in (-\infty, x_0] \\ \phi_0 - \frac{\pi}{2} - k_0(x'_1 - x_0), & x'_1 \in (x_0, x_0 + a] \\ \phi_0 + \frac{\pi}{2}, & x'_1 \in (x_0 + a, \infty) \end{cases}. \quad (210)$$

$x'_1 = \vec{x} \cdot \vec{e}$ is the coordination on the axis along a given direction \vec{e} and ϕ_0 is a constant angle, $k_0 = \frac{\pi}{a}$. $+$ denotes clockwise winding (knot with up-spin) and $-$ anti-clockwise winding (knot with down-spin).

We may generate a linear knot by doing knot-operation on chiral knot-crystal

$$\begin{aligned} \mathbf{Z}_{\text{total}}(\vec{x}, t) &= \begin{pmatrix} 0 \\ z_B(\vec{x}, t) \end{pmatrix}_{\text{total}} \\ &= \mathbf{U}_{\text{knot}} \mathbf{Z}(\vec{x}, t) \end{aligned} \quad (211)$$

where

$$\begin{aligned} \mathbf{U}_{\text{knot}} &= \exp[\pm i\phi_{\text{knot}, B}(x'_1) \cdot \hat{K}], \\ \phi_{\text{knot}, B}(x'_1) &= \begin{cases} -\frac{\pi}{2}, & x'_1 \in (-\infty, x_0] \\ -\frac{\pi}{2} \pm k_0(x'_1 - x_0), & x'_1 \in (x_0, x_0 + a] \\ \frac{\pi}{2}, & x'_1 \in (x_0 + a, \infty) \end{cases}, \end{aligned}$$

where $\hat{K} = -i\frac{d}{d\phi_B}$ is knot-number operator and $x'_1 = \vec{x} \cdot \vec{e}$ is variable along \vec{e} direction.

As a result, a linear knot is piece of knot-crystal and becomes elementary excitation in it.

b. 1-level double-helix knot-crystal It is known that for a 1-level double-helix knot-crystal with $\mathcal{N} = 2$, there are four possible chiralities, i.e.,

$$\mathbf{Z}_{\text{LL}}(\vec{e}, \vec{x}, t) = \begin{pmatrix} z_{A, L}(\vec{x}, t) \\ z_{B, L}(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} r_A e^{i\vec{k}_0 \cdot \vec{x} + i\omega_A t + i\phi_A} \\ r_B e^{i\vec{k}_0 \cdot \vec{x} + i\omega_B t + i\phi_B} \end{pmatrix}, \quad (212)$$

$$\mathbf{Z}_{\text{LR}}(\vec{e}, \vec{x}, t) = \begin{pmatrix} z_A(\vec{x}, t) \\ z_B(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} r_A e^{i\vec{k}_0 \cdot \vec{x} + i\omega_A t + i\phi_A} \\ r_B e^{-i\vec{k}_0 \cdot \vec{x} + i\omega_B t + i\phi_B} \end{pmatrix}, \quad (213)$$

$$\mathbf{Z}_{\text{RL}}(\vec{e}, \vec{x}, t) = \begin{pmatrix} z_A(\vec{x}, t) \\ z_B(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} r_A e^{-i\vec{k}_0 \cdot \vec{x} + i\omega_A t + i\phi_A} \\ r_B e^{i\vec{k}_0 \cdot \vec{x} + i\omega_B t + i\phi_B} \end{pmatrix}, \quad (214)$$

$$\mathbf{Z}_{\text{RR}}(\vec{e}, \vec{x}, t) = \begin{pmatrix} z_A(\vec{x}, t) \\ z_B(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} r_A e^{-i\vec{k}_0 \cdot \vec{x} + i\omega_A t + i\phi_A} \\ r_B e^{-i\vec{k}_0 \cdot \vec{x} + i\omega_B t + i\phi_B} \end{pmatrix}, \quad (215)$$

where ϕ_A and ϕ_B are constant angles, ω_A and ω_B are rotating velocities. r_A and r_B are the winding radii of brane-A and brane-B in extra dimensional space. In general, r_A and r_B are different $r_A \neq r_B$ but very close, $r_A \simeq r_B$.

Knot-crystal-operation for 1-level double-helix knot-crystal For the four types of $\mathcal{N} = 2$ static 1-level double-helix knot-crystals, $\mathbf{Z}_{\text{LL}}(\vec{e}, \vec{x})$, $\mathbf{Z}_{\text{LR}}(\vec{e}, \vec{x})$, $\mathbf{Z}_{\text{RL}}(\vec{e}, \vec{x})$, $\mathbf{Z}_{\text{RR}}(\vec{e}, \vec{x})$, we may define four knot-crystal-operations

$$\begin{aligned} \hat{U}(\mathbf{Z}_{\text{LL}}) &= \hat{U}(\mathbf{z}_A) \otimes \hat{U}(\mathbf{z}_B), \\ \hat{U}(\mathbf{Z}_{\text{LR}}) &= \hat{U}(\mathbf{z}_A) \otimes \hat{U}^*(\mathbf{z}_B), \\ \hat{U}(\mathbf{Z}_{\text{RL}}) &= \hat{U}^*(\mathbf{z}_A) \otimes \hat{U}(\mathbf{z}_B), \\ \hat{U}(\mathbf{Z}_{\text{RR}}) &= \hat{U}^*(\mathbf{z}_A) \otimes \hat{U}^*(\mathbf{z}_B), \end{aligned} \quad (216)$$

with

$$\begin{aligned}\hat{U}(\mathbf{z}_A) &= \exp[i\phi_A(\vec{x}) \cdot \hat{K}], \\ \hat{U}(\mathbf{z}_B) &= \exp[i\phi_B(\vec{x}) \cdot \hat{K}],\end{aligned}\tag{217}$$

where $\phi_A(\vec{x}, t) = \vec{k}_0 \cdot \vec{x} + \phi_A$ and $\phi_B(\vec{x}, t) = \vec{k}_0 \cdot \vec{x} + \phi_B$. Thus, we have

$$\begin{aligned}\mathbf{Z}_{LL}(\vec{e}, \vec{x}) &= \hat{U}(\mathbf{Z}_{LL}) \begin{pmatrix} r_A \\ r_B \end{pmatrix}, \\ \mathbf{Z}_{LR}(\vec{e}, \vec{x}) &= \hat{U}(\mathbf{Z}_{LR}) \begin{pmatrix} r_A \\ r_B \end{pmatrix}, \\ \mathbf{Z}_{RL}(\vec{e}, \vec{x}) &= \hat{U}(\mathbf{Z}_{RL}) \begin{pmatrix} r_A \\ r_B \end{pmatrix}, \\ \mathbf{Z}_{RR}(\vec{e}, \vec{x}) &= \hat{U}(\mathbf{Z}_{RR}) \begin{pmatrix} r_A \\ r_B \end{pmatrix}.\end{aligned}\tag{218}$$

Under parity transformation in extra space, we have

$$\begin{aligned}\mathbf{Z}_{LL}(\vec{e}, \vec{x}) &= \hat{\mathcal{P}}(x_{d+1} - x_{d+2}) \mathbf{Z}_{RR}(\vec{e}, \vec{x}) \\ &= \mathbf{Z}_{RR}^*(\vec{e}, \vec{x}), \\ \mathbf{Z}_{LR}(\vec{e}, \vec{x}) &= \hat{\mathcal{P}}(x_{d+1} - x_{d+2}) \mathbf{Z}_{RL}(\vec{e}, \vec{x}) \\ &= \mathbf{Z}_{RL}^*(\vec{e}, \vec{x}).\end{aligned}\tag{219}$$

There are two different types of 1-level double-helix knot-crystals with $\mathcal{N} = 2$ – one is $\mathbf{Z}_{LL}(\vec{e}, \vec{x}) = \mathbf{Z}_{RR}^*(\vec{e}, \vec{x})$, the other is $\mathbf{Z}_{LR}(\vec{e}, \vec{x}) = \mathbf{Z}_{RL}^*(\vec{e}, \vec{x})$. We call $\mathbf{Z}_{LL}(\vec{e}, \vec{x})$ or $\mathbf{Z}_{RR}(\vec{e}, \vec{x})$ to be symmetric double-helix knot-crystal and $\mathbf{Z}_{LR}(\vec{e}, \vec{x})$ or $\mathbf{Z}_{RL}(\vec{e}, \vec{x})$ to be anti-symmetric double-helix knot-crystal.

For a symmetric double-helix knot-crystal with $\mathbf{Z}_{LL}(\vec{e}, \vec{x})$ or $\mathbf{Z}_{RR}(\vec{e}, \vec{x})$, there are two types of brane-twisting motions along a given direction \vec{e} on brane – one can move a knot by twisting brane-A clockwise or by twisting brane-B anti-clockwise. For a anti-symmetric double-helix knot-crystal with $\mathbf{Z}_{LR}(\vec{e}, \vec{x})$ or $\mathbf{Z}_{RL}(\vec{e}, \vec{x})$, there are also two types of brane-twisting motions along a given direction \vec{e} on brane – one can move a knot by twisting brane-A clockwise or by twisting brane-B clockwise.

Static 1-level double-helix knot-crystal For static double-helix knot-crystal, the knot-function is defined by

$$\mathbf{Z}(\vec{e}, \vec{x}) = \begin{pmatrix} z_A(\vec{x}) \\ z_B(\vec{x}) \end{pmatrix} = \begin{pmatrix} r_A e^{i[\pm \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_A]} \\ r_B e^{i[\pm \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_B]} \end{pmatrix}\tag{220}$$

In Fig.12, we plot the phase vs x for $\mathbf{Z}_{LL}(\vec{e}, \vec{x}, t)$ and a symmetric 1-level double-helix knot-crystal is denoted by two lines, of which the slope of A/B brane is $\frac{\pi}{a}$.

Due to $r_A \simeq r_B$, we may consider $r_A = r_B = r$. We then solve the knot-equation $\hat{P}_\theta[z_A(x)] = \hat{P}_\theta[z_B(x)]$ for the four types of static double-helix knot-crystals:

1. For $\mathbf{Z}_{LL}(\vec{e}, \vec{x})$, the knot-equation $\hat{P}_\theta[z_A(x)] = \hat{P}_\theta[z_B(x)]$ becomes

$$\cos\left[\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_A\right] = \cos\left[\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_B\right].\tag{221}$$

The knot-solution is given by

$$\left[\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_A\right] \pm \left[\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_B\right] = 2\pi n\tag{222}$$

or

$$\vec{x}' = a \cdot n - \frac{a}{2\pi}(\phi_A + \phi_B);\tag{223}$$

2. For $\mathbf{Z}_{\text{LR}}(\vec{e}, \vec{x})$, the knot-equation $\hat{P}_\theta[z_A(x)] = \hat{P}_\theta[z_B(x)]$ becomes

$$\cos\left[\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_A\right] = \cos\left[-\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_B\right]. \quad (224)$$

The knot-solution is given by

$$\left[\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_A\right] \pm \left[-\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_B\right] = 2\pi n \quad (225)$$

or

$$\vec{x}' = a \cdot n - \frac{a}{2\pi}(\phi_A - \phi_B); \quad (226)$$

3. For $\mathbf{Z}_{\text{RL}}(\vec{e}, \vec{x})$, the knot-equation $\hat{P}_\theta[z_A(x)] = \hat{P}_\theta[z_B(x)]$ becomes

$$\cos\left[-\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_A\right] = \cos\left[\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_B\right]. \quad (227)$$

The knot-solution is given by

$$\left[-\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_A\right] \pm \left[\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_B\right] = 2\pi n \quad (228)$$

or

$$\vec{x}' = -a \cdot n + \frac{a}{2\pi}(-\phi_A + \phi_B); \quad (229)$$

4. For $\mathbf{Z}_{\text{RR}}(\vec{e}, \vec{x})$, the knot-equation $\hat{P}_\theta[z_A(x)] = \hat{P}_\theta[z_B(x)]$ becomes

$$\cos\left[-\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_A\right] = \cos\left[-\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_B\right]. \quad (230)$$

The knot-solution is given by

$$\left[-\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_A\right] \pm \left[-\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_B\right] = 2\pi n \quad (231)$$

or

$$\vec{x}' = -a \cdot n + \frac{a}{2\pi}(\phi_A + \phi_B). \quad (232)$$

Rotating 1-level double-helix knot-crystal In this section, we consider rotating double-helix knot-crystal. For a rotating double-helix knot-crystal, the knot-function is defined by

$$\mathbf{Z}(\vec{e}, \vec{x}) = \begin{pmatrix} z_A(\vec{x}, t) \\ z_B(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} r_A e^{i[\pm \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \omega_A t + \phi_A]} \\ r_B e^{i[\pm \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \omega_B t + \phi_B]} \end{pmatrix} \quad (233)$$

For a rotating double-helix knot-crystal, the knot-solution ($\hat{P}_\theta[z_A(\vec{x}, t)] = \hat{P}_\theta[z_B(\vec{x}, t)]$) becomes

$$\begin{aligned} 2\pi n &= [\pm \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \omega_A t + \phi_A] \\ &\quad \pm [\pm \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \omega_B t + \phi_B]. \end{aligned} \quad (234)$$

We then solve the knot-equation for the four types of static double-helix knot-crystals:

1. For $\mathbf{Z}_{\text{LL}}(\vec{e}, \vec{x}, t)$, the knot-solution is given by

$$\vec{x}' = a \cdot n - \frac{a}{2\pi}(\omega_A + \omega_B)t - \frac{a}{2\pi}(\phi_A + \phi_B); \quad (235)$$

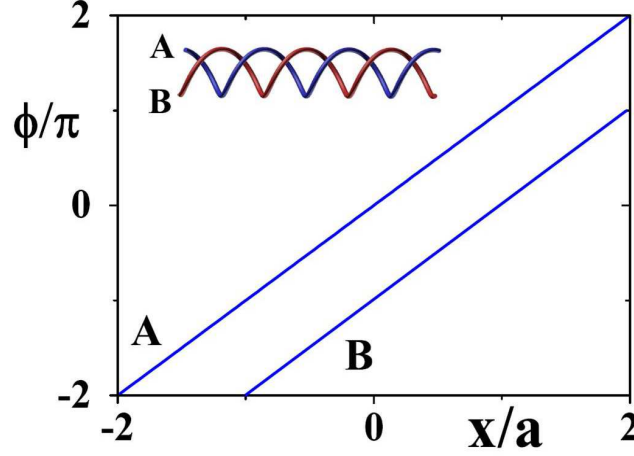


FIG. 12: The phase vs x of a 1D double-helix knot-crystal. The scale of x is $\frac{1}{a}$. A and B denote the indices of 1-branes. The inset is an illustration of 1D double-helix knot-crystal.

2. For $\mathbf{Z}_{LR}(\vec{e}, \vec{x}, t)$, the knot-solution is given by

$$\vec{x}' = a \cdot n - \frac{a}{2\pi}(\omega_A - \omega_B)t - \frac{a}{2\pi}(\phi_A - \phi_B); \quad (236)$$

3. For $\mathbf{Z}_{RL}(\vec{e}, \vec{x}, t)$, the knot-solution is given by

$$\vec{x}' = -a \cdot n - \frac{a}{2\pi}(-\omega_A + \omega_B)t - \frac{a}{2\pi}(-\phi_A + \phi_B); \quad (237)$$

4. For $\mathbf{Z}_{RR}(\vec{e}, \vec{x}, t)$, the knot-solution is given by

$$\vec{x}' = -a \cdot n - \frac{a}{2\pi}(-\omega_A - \omega_B)t - \frac{a}{2\pi}(-\phi_A - \phi_B). \quad (238)$$

We introduce the concept of "balancedly-rotating knot-crystal" and the concept of "unbalancedly-rotating knot-crystal".

A balancedly-rotating knot-crystal is a rotating knot-crystal with static knot-solutions. For $\mathbf{Z}_{LL}(\vec{e}, \vec{x}, t)$ and $\mathbf{Z}_{RR}(\vec{e}, \vec{x}, t)$, we get balancedly-rotating knot-crystal for the case of $\omega_A + \omega_B = 0$ (or $\omega_A = -\omega_B = \omega \neq 0$); for $\mathbf{Z}_{LR}(\vec{e}, \vec{x}, t)$ and $\mathbf{Z}_{RL}(\vec{e}, \vec{x}, t)$, we get balancedly-rotating knot-crystal for the case of $\omega_A - \omega_B = 0$ (or $\omega_A = \omega_B = \omega \neq 0$). An unbalancedly-rotating knot-crystal is a rotating knot-crystal with moving knot-solutions. For $\mathbf{Z}_{LL}(\vec{e}, \vec{x}, t)$ and $\mathbf{Z}_{RR}(\vec{e}, \vec{x}, t)$, we get unbalancedly-rotating knot-crystal for the case of $\omega_A + \omega_B \neq 0$ (or $\omega_A \neq -\omega_B$); for $\mathbf{Z}_{LR}(\vec{e}, \vec{x}, t)$ and $\mathbf{Z}_{RL}(\vec{e}, \vec{x}, t)$, we get unbalancedly-rotating knot-crystal for the case of $\omega_A - \omega_B \neq 0$ (or $\omega_A \neq \omega_B$).

It was known that there are always *two possible* knot-solutions at a point from out-plane projection: for one solution, we have $\eta_{A,\theta}(x'_1) > \eta_{B,\theta}(x'_1)$ or $\text{sgn}[\Upsilon] = \text{sgn}[\eta_{A,\theta}(x'_1, t) - \eta_{B,\theta}(x'_1, t)] = 1$; for the other solution, we have $\eta_{A,\theta}(x'_1) < \eta_{B,\theta}(x'_1)$ or $\text{sgn}[\Upsilon] = \text{sgn}[\eta_{A,\theta}(x'_1, t) - \eta_{B,\theta}(x'_1, t)] = -1$. To characterize balancedly-rotating knot-crystal, we introduce a variable – the sign of Υ ,

$$\text{sgn}[\Upsilon] = \text{sgn}[\eta_{A,\theta}(x'_1, t) - \eta_{B,\theta}(x'_1, t)] \quad (239)$$

where

$$\begin{aligned} \eta_{A,\theta}(x'_1, t) &= \sin\left[\pm \frac{\pi}{a}x'_1 + \omega_A t + \phi_A\right], \\ \eta_{B,\theta}(x'_1, t) &= \sin\left[\pm \frac{\pi}{a}x'_1 + \omega_B t + \phi_B\right]. \end{aligned} \quad (240)$$

For a balancedly-rotating knot-crystal, we have

$$\text{sgn}[\Upsilon(t)] = -\text{sgn}[\Upsilon(t + \frac{\pi}{\omega})]. \quad (241)$$

For a balancedly-rotating knot-crystal, all knots are static at given position with periodically time-changing of the sign of $\Upsilon(t)$.

Thus, for $\mathbf{Z}_{LL}(\vec{e}, \vec{x}, t)$ and $\mathbf{Z}_{RR}(\vec{e}, \vec{x}, t)$ with $\omega_A = -\omega_B = \omega$, the sign of

$$\begin{aligned}\Upsilon &= \sin[\omega t + \phi_A] - \sin[-\omega t + \phi_B] \\ &= \sin \omega t (\cos \phi_A + \cos \phi_B) \\ &\quad + \cos \omega t (\sin \phi_A - \sin \phi_B)\end{aligned}\tag{242}$$

is periodically changed via time

$$\text{sgn}[\Upsilon(t)] = -\text{sgn}[\Upsilon(t + \frac{\pi}{\omega})].\tag{243}$$

For $\mathbf{Z}_{LR}(\vec{e}, \vec{x}, t)$ and $\mathbf{Z}_{RL}(\vec{e}, \vec{x}, t)$ with $\omega_A = \omega_B = \omega$, the sign of

$$\begin{aligned}\Upsilon &= \sin[\omega t + \phi_A] - \sin[\omega t + \phi_B] \\ &= \sin \omega t (\cos \phi_A - \cos \phi_B) \\ &\quad + \cos \omega t (\sin \phi_A - \sin \phi_B)\end{aligned}\tag{244}$$

is also periodically changed via time

$$\text{sgn}[\Upsilon(t)] = -\text{sgn}[\Upsilon(t + \frac{\pi}{\omega})].\tag{245}$$

Linear knot on double-helix knot-crystal There are two types of linear knots on double-helix knot-crystal, A-type and B-type.

A-type linear knot is a half-winding of d -brane-A around d -brane-B, of which the knot-function is given by

$$\mathbf{Z}_{\text{knot}}(\vec{x}, t) = \begin{pmatrix} z_A(\vec{x}, t) \\ z_B(\vec{x}, t) \end{pmatrix}_{\text{knot}} = \begin{pmatrix} r_A \exp[\pm i \phi_{\text{knot}, A}(x)] \\ r_B \end{pmatrix}\tag{246}$$

where

$$\phi_{\text{knot}, A}(x'_1) = \begin{cases} \phi_0 - \frac{\pi}{2}, & x'_1 \in (-\infty, x_0] \\ \phi_0 - \frac{\pi}{2} - k_0(x'_1 - x_0), & x'_1 \in (x_0, x_0 + a] \\ \phi_0 + \frac{\pi}{2}, & x'_1 \in (x_0 + a, \infty) \end{cases}.\tag{247}$$

$x'_1 = \vec{x} \cdot \vec{e}$ is the coordination on the axis along a given direction \vec{e} and ϕ_0 is a constant angle, $k_0 = \frac{\pi}{a}$. $+$ denotes clockwise winding (knot with up-spin) and $-$ anti-clockwise winding (knot with down-spin).

B-type linear knot is a half-winding of d -brane-B around d -brane-A, of which the knot-function is given by

$$\mathbf{Z}_{\text{knot}}(\vec{x}, t) = \begin{pmatrix} z_A(\vec{x}, t) \\ z_B(\vec{x}, t) \end{pmatrix}_{\text{knot}} = \begin{pmatrix} r_A \\ r_B \exp[\pm i \phi_{\text{knot}, B}(x)] \end{pmatrix}\tag{248}$$

where

$$\phi_{\text{knot}, B}(x'_1) = \begin{cases} \phi_0 - \frac{\pi}{2}, & x'_1 \in (-\infty, x_0] \\ \phi_0 - \frac{\pi}{2} - k_0(x'_1 - x_0), & x'_1 \in (x_0, x_0 + a] \\ \phi_0 + \frac{\pi}{2}, & x'_1 \in (x_0 + a, \infty) \end{cases}.\tag{249}$$

We may also generate a linear knot by doing knot-operation on double-helix knot-crystal

$$\begin{aligned}\mathbf{Z}_{\text{total}, A/B}(\vec{x}, t) &= \begin{pmatrix} z_A(\vec{x}, t) \\ z_B(\vec{x}, t) \end{pmatrix}_{\text{total}} \\ &= \mathbf{U}_{\text{knot}, A/B} \mathbf{Z}_{L/R}(\vec{x}, t)\end{aligned}\tag{250}$$

where

$$\begin{aligned}\mathbf{U}_{\text{knot}} &= \exp[\pm i \phi_{\text{knot}, A/B}(x'_1) \cdot \hat{K}], \\ \phi_{\text{knot}, A/B}(x'_1) &= \begin{cases} -\frac{\pi}{2}, & x'_1 \in (-\infty, x_0] \\ -\frac{\pi}{2} - k_0(x'_1 - x_0), & x'_1 \in (x_0, x_0 + a] \\ \frac{\pi}{2}, & x'_1 \in (x_0 + a, \infty) \end{cases},\end{aligned}$$

where $\hat{K} = -i \frac{d}{d\phi_{A/B}}$.

As a result, a linear knot on double-helix knot-crystal has four-component: two come from spin (winding direction) degrees of freedom (spin-up and spin-down), the other two from chiral (brane) degrees of freedom (A-type and B-type or left-hand and right-hand).

2. Generalized translation symmetry of 1-level knot-crystal

a. Discrete spatial translation symmetry in solid state Symmetry and symmetry breaking play profound roles in many-body physics and particle physics.

The eigenstates of an arbitrary quantum state in quantum field theory have continuous spatial translation symmetry $|\vec{k}_{\text{continuous}}\rangle$ where $\vec{k}_{\text{continuous}}$ is wave-vector. By defining a translation operation $T(\Delta\vec{x})$, we have

$$T(\Delta\vec{x})|\vec{k}_{\text{continuous}}\rangle = e^{i\vec{k}_{\text{continuous}}\cdot\Delta\vec{x}}|\vec{k}_{\text{continuous}}\rangle \quad (251)$$

where $\Delta\vec{x}$ denotes the distance from spatial translation operation.

In solid physics, a crystal is the system of periodically arranged atoms with spontaneous breaking of continuous spatial translation symmetry to discrete spatial translation symmetry. On a crystal, due to spontaneous breaking of continuous spatial translation symmetry to discrete spatial translation symmetry, the eigenstates are quantum states $|\vec{k}_{\text{discrete}}\rangle$. The Bloch vector $|\vec{k}_{\text{discrete}}\rangle$ is defined as a linear superposition of the localized states $|\vec{R}, \vec{a}_j\rangle$ in the unit cell \vec{R} at position \vec{a}_j ,

$$|\vec{k}_{\text{discrete}}\rangle = \sum_{\vec{R}, j} c_j(\vec{k}) e^{i\vec{k}_{\text{discrete}}\cdot\vec{x}} |\vec{R}, \vec{a}_j\rangle, \quad (252)$$

that depends on Bloch momentum $\vec{k}_{\text{discrete}}$ and type of orbital j in the unit cell. One cannot do a translation operation $T(\Delta\vec{x})$ with $\Delta\vec{x} = \alpha\vec{R}$ where α is a non-integer real number.

The Bloch theorem is very useful in description of a quasi-particle in crystals that states that the basis vector $|\vec{k}_{\text{discrete}}\rangle$ translated for the Bravais vector \vec{R} changes as

$$T(\vec{R})|\vec{k}_{\text{discrete}}\rangle = e^{i\vec{k}_{\text{discrete}}\cdot\vec{R}}|\vec{k}\rangle \quad (253)$$

where $T(\vec{R})$ is translation operation. Due to the existence of Brillouin zone (BZ), we have $|\vec{k}_{\text{discrete}}\rangle = |\vec{k}_{\text{discrete}} + \vec{Q}\rangle$ where \vec{Q} is reciprocal lattice vector.

b. Generalized translation symmetry for knot-crystal We use knot-crystal-operation to discuss the generalized translation symmetry of knot-crystal.

Firstly, for two types of $\mathcal{N} = 1$ 1-level knot-crystals, $\mathbf{Z}_L(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_R(\vec{e}, \vec{x}, t)$, we have defined two knot-crystal-operations $\hat{U}(\mathbf{z}_A)$ and $\hat{U}(\mathbf{z}_B)$ that are

$$\begin{aligned} \hat{U}(\mathbf{z}_A(\vec{x}, t)) &= \exp[i\phi_A(\vec{x}, t) \cdot \hat{K}], \\ \hat{U}(\mathbf{z}_B(\vec{x}, t)) &= \exp[i\phi_B(\vec{x}, t) \cdot \hat{K}], \end{aligned} \quad (254)$$

where $\phi_A(\vec{x}, t) = \vec{k}_0 \cdot \vec{x} + \omega_A t + \phi_A$ and $\phi_B(\vec{x}, t) = \vec{k}_0 \cdot \vec{x} + \omega_B t + \phi_B$. The generalized continuous symmetry of $\mathcal{N} = 1$ 1-level knot-crystal is obtained by

$$\mathcal{T}(\Delta\vec{x})\hat{U}(\mathbf{z}_A)\mathcal{T}^{-1}(\Delta\vec{x}) = \hat{U}(\mathbf{z}_A(\vec{x} + \Delta\vec{x}, t)),$$

and

$$\mathcal{T}(\Delta\vec{x})\hat{U}(\mathbf{z}_B)\mathcal{T}^{-1}(\Delta\vec{x}) = \hat{U}(\mathbf{z}_B(\vec{x} + \Delta\vec{x}, t)).$$

Thus, for 1-level chiral knot-crystal with $z_i = r_i \exp(i\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \phi_{0,i})$, there exists a generalized translation symmetry, i.e.,

$$\begin{aligned} \mathcal{T}(\Delta\vec{x})\mathbf{z}_A &= \exp[i\frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x})]\mathbf{z}_A, \\ \mathcal{T}(\Delta\vec{x})\mathbf{z}_B &= \exp[i\frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x})]\mathbf{z}_B. \end{aligned} \quad (255)$$

Using similar approach, for 1-level double-helix knot-crystal, the generalized translation symmetry is obtained as

$$\begin{aligned} \mathcal{T}(\Delta\vec{x})\mathbf{Z}_{LL}(\vec{e}, \vec{x}, t) &= \mathcal{T}(\Delta\vec{x}) \begin{pmatrix} r_A e^{i\vec{k}_0 \cdot \vec{x} + i\phi_A} \\ r_B e^{i\vec{k}_0 \cdot \vec{x} + i\phi_B} \end{pmatrix} \\ &= \exp[i\frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x} \cdot \mathbf{1})]\mathbf{Z}_{LL}(\vec{e}, \vec{x}, t), \end{aligned} \quad (256)$$

$$\begin{aligned}
\mathcal{T}(\Delta\vec{x})\mathbf{Z}_{\text{RL}}(\vec{e}, \vec{x}, t) &= \mathcal{T}(\Delta\vec{x}) \begin{pmatrix} r_{\text{A}} e^{i\vec{k}_0 \cdot \vec{x} + i\phi_{\text{A}}} \\ r_{\text{B}} e^{-i\vec{k}_0 \cdot \vec{x} + i\phi_{\text{B}}} \end{pmatrix} \\
&= \exp\left[i\frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x} \cdot \tau^z)\right] \mathbf{Z}_{\text{RL}}(\vec{e}, \vec{x}, t),
\end{aligned} \tag{257}$$

$$\begin{aligned}
\mathcal{T}(\Delta\vec{x})\mathbf{Z}_{\text{LR}}(\vec{e}, \vec{x}, t) &= \mathcal{T}(\Delta\vec{x}) \begin{pmatrix} r_{\text{A}} e^{-i\vec{k}_0 \cdot \vec{x} + i\phi_{\text{A}}} \\ r_{\text{B}} e^{i\vec{k}_0 \cdot \vec{x} + i\phi_{\text{B}}} \end{pmatrix} \\
&= \exp\left[-i\frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x} \cdot \tau^z)\right] \mathbf{Z}_{\text{LR}}(\vec{e}, \vec{x}, t),
\end{aligned} \tag{258}$$

$$\begin{aligned}
\mathcal{T}(\Delta\vec{x})\mathbf{Z}_{\text{RR}}(\vec{e}, \vec{x}, t) &= \mathcal{T}(\Delta\vec{x}) \begin{pmatrix} r_{\text{A}} e^{i\vec{k}_0 \cdot \vec{x} + i\phi_{\text{A}}} \\ r_{\text{B}} e^{i\vec{k}_0 \cdot \vec{x} + i\phi_{\text{B}}} \end{pmatrix} \\
&= \exp\left[-i\frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x} \cdot \mathbf{1})\right] \mathbf{Z}_{\text{RR}}(\vec{e}, \vec{x}, t).
\end{aligned} \tag{259}$$

c. Generalized Bloch theorem for quantum states of 1-level knot-crystal In this work we present a generalization of the Bloch theorem that incorporates generalized translational symmetries for quantum states of a 1-level knot-crystal.

For a quantum plane-wave state $|\vec{k}\rangle = |\vec{k}_0/N_k\rangle$ on 1-level knot-crystal (that is a waving brane-twisting pattern with integer number N_k), we do a translation operation $\hat{T}(\Delta\vec{x})$ (that corresponds to $\mathcal{T}(\Delta\vec{x})$ on knot-crystal) and get

$$\hat{T}(\Delta\vec{x}) |\vec{k}\rangle = e^{\pm i\vec{k}_0 \cdot [2a[(\Delta\vec{x}) \bmod 2a]]} e^{i\vec{k} \cdot [(\Delta\vec{x}) - 2a[(\Delta\vec{x}) \bmod 2a]]} |\vec{k}\rangle \tag{260}$$

where $(\Delta\vec{x}) - 2a[(\Delta\vec{x}) \bmod 2a]$ denotes an integer winding number along the direction \vec{k} . Thus, an arbitrary continuous spatial translation operation is combination of a discrete spatial translation

$$\begin{aligned}
|\vec{k}\rangle &\rightarrow |\vec{k}'\rangle = \hat{T}(\Delta\vec{x} = \vec{e}(2aN)) |\vec{k}\rangle \\
&= e^{i\vec{k} \cdot \vec{e}(2aN)} |\vec{k}\rangle
\end{aligned} \tag{261}$$

and a global gauge transformation

$$\begin{aligned}
|\vec{k}'\rangle &\rightarrow |\vec{k}\rangle = U(\Delta\phi) |\vec{k}\rangle \\
&= e^{\pm i\vec{k}_0 \cdot [2a[(\Delta\vec{x}) \bmod 2a]]} |\vec{k}\rangle
\end{aligned} \tag{262}$$

by choosing different global phases. Here $U(\Delta\phi) = e^{i\Delta\phi} = e^{\pm i\vec{k}_0 \cdot [2a[(\Delta\vec{x}) \bmod 2a]]}$ is the operator of phase transformation.

For the case of $(\Delta\vec{x}) \bmod 2a = 0$, we have

$$\hat{T}(\Delta\vec{x}) |\vec{k}\rangle = e^{i\vec{k} \cdot (\Delta\vec{x})} |\vec{k}\rangle \tag{263}$$

that is reduced to the results in solid state physics. And we have a generalized BZ along a given direction \vec{e} as

$$|\vec{k}\rangle = |\vec{k} + \vec{k}_0\rangle \tag{264}$$

where $\vec{k}_0 = \frac{\pi}{a}\vec{e}$ is reciprocal lattice vector along the given direction \vec{e} . The generalized Bloch theorem constrains the formulation of the Hamiltonian which becomes manifestly invariant under additional spacial rotating symmetry.

For the case of $(\Delta\vec{x}) \bmod 2a \neq 0$, we can recover the continuous spatial translation symmetry by changing projection angle

$$\begin{aligned}
\theta &\rightarrow \theta' = \theta \mp k_0 \cdot [2a[(\Delta\vec{x}) \bmod 2a]] \\
&= \theta \mp 2\pi[(\Delta\vec{x}) \bmod 2a].
\end{aligned} \tag{265}$$

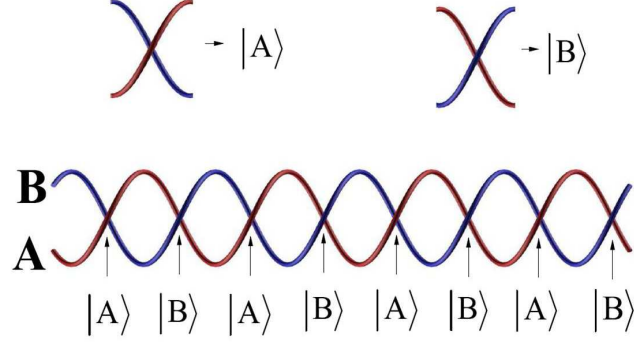


FIG. 13: An illustration of two "sublattices" of 1-level double-helix knot-crystal: $|A\rangle$ and $|B\rangle$ for the 1D double-helix knot-crystal. A and B denote the indices of 1-branes.

3. Emergent quantum field theory

Knot mechanics is a mathematical description of the dynamic entanglement pattern between two branes. In a sense, we can view a knot-crystal as a particular static entangled pattern in which a brane winds around each other. In this part, we regard Dirac (double-helix) knot-crystal as a multi-knot system and knots become elementary fermionic particles in a Dirac knot-crystal. From this point of view, the quantum fluctuations of 3D balancedly-rotating Dirac knot-crystals correspond to quantum field theory of free massive Dirac fermions. The four types of 3D Dirac knot-crystals with different chiralities have the same effective Dirac model for free massive Dirac fermions.

a. Quantum states of 3D 1-level balancedly-rotating knot-crystal Firstly, we show the quantum states of a 3D 1-level balancedly-rotating Dirac knot-crystal. It was known that there are always *two possible* knot-solutions at a point from out-plane projection: a solution with $\eta_{A,\theta}(x'_1) > \eta_{B,\theta}(x'_1)$ or $\text{sgn}[\Upsilon] = \text{sgn}[\eta_{A,\theta}(x'_1, t) - \eta_{B,\theta}(x'_1, t)] = 1$ and the other solution with $\eta_{A,\theta}(x'_1) < \eta_{B,\theta}(x'_1)$ or $\text{sgn}[\Upsilon] = \text{sgn}[\eta_{A,\theta}(x'_1, t) - \eta_{B,\theta}(x'_1, t)] = -1$. These two types of degrees of freedom correspond to chiral degrees of freedom in knot-crystal theory. We define the states of knot-solution $\text{sgn}[\Upsilon] = 1$ by $|A\rangle$ and the states of knot-solution $\text{sgn}[\Upsilon] = -1$ by $|B\rangle$, respectively. In Fig.13, we consider a Dirac knot-crystal to be a "two-sublattice" model with discrete spatial translation symmetry. We then label the knots by quantum state $|x'_1, A\rangle, |x'_1 + a, B\rangle, |x'_1 + 2a, A\rangle, \dots$

b. Low energy effective Lagrangian of 3D 1-level balancedly-rotating Dirac knot-crystal For a 3D 1-level balancedly-rotating Dirac knot-crystal, there exist two types of energy costs – energy cost from curving a brane and that from rotating the Dirac knot-crystal. To characterize the energy cost from brane-curving, we define the coupling constant J between two nearest-neighbor knots by curving the branes. On the other hand, to characterize the energy cost from balancedly-rotating of Dirac knot-crystal, we assume the rotating velocity to be $\omega = \omega_A = -\omega_B$ for $\mathbf{Z}_{LL}(\vec{e}, \vec{x}, t)$ and $\mathbf{Z}_{RR}(\vec{e}, \vec{x}, t)$ and $\omega_A = \omega_B = \omega$ for $\mathbf{Z}_{LR}(\vec{e}, \vec{x}, t)$ and $\mathbf{Z}_{RL}(\vec{e}, \vec{x}, t)$.

We firstly study the energy cost from curving both A-brane and B-brane. From Fig.13, there are two links that connects two knots at $|A\rangle$ and $|B\rangle$. We use the following formulation to characterize the energy from brane-curving that connects two nearest neighbor states $|x'_1, A\rangle$ and $|x'_1 + a, B\rangle$,

$$\begin{aligned} & \frac{J}{2} (|x'_1, A\rangle \langle x'_1 + a, B| + |x'_1 + a, B\rangle \langle x'_1, A|) \\ & = \frac{J}{2} (|x'_1, A\rangle \langle x'_1 + a, B| + h.c.). \end{aligned} \quad (266)$$

According to above formulation, we use an effective Hamiltonian to describe the dynamics of balancedly-rotating knot-crystal

$$\frac{J}{2} \sum_{\langle i, j \rangle} (|i, A\rangle \langle j, B| + h.c.) \quad (267)$$

where $|i, A\rangle = |x'_1 + i \cdot a, A\rangle$ and i denotes a given knot at the position $x'_1 + i \cdot a$. $\langle i, j \rangle$ denotes the nearest-neighbor knots. With help of translation operator $\hat{T}(a)$, we have

$$|x'_1 + a, A\rangle = \hat{T}(a) \cdot \hat{S} |x'_1, B\rangle \quad (268)$$

where $\hat{T}(a)$ is the translation operator $\hat{T}(a) |x'_1\rangle = |x'_1 + a\rangle$ that is

$$\hat{T}(a) = e^{ia(\vec{k} \cdot \vec{\sigma})} \quad (269)$$

and \hat{S} is the chiral translation operator that is

$$\hat{S} = (A \rightarrow B). \quad (270)$$

The effective Hamiltonian from brane-curling can be described by a familiar formulation

$$\hat{\mathcal{H}}_{\text{coupling}} = \frac{J}{2} \sum_{\langle i, j \rangle} c_{A,i}^\dagger c_{B,j} + h.c. \quad (271)$$

where \hat{c}_i^\dagger is the annihilation operator of knots at the site i . We then use path-integral formulation to characterize the effective Hamiltonian for a knot-crystal as

$$\int \mathcal{D}\psi^\dagger(t, \vec{x}) \mathcal{D}\psi(t) e^{iS/\hbar} \quad (272)$$

where $S = \int \mathcal{L} dt$ and $\mathcal{L} = i \sum_i \psi_i^\dagger \partial_t \psi_i - \mathcal{H}_{\text{coupling}}$. To describe the 3D balancedly-rotating Dirac knot-crystal, we have introduced a four-component fermion field to be

$$\psi(\mathbf{x}) = \begin{pmatrix} \psi_{A\uparrow}(t, \vec{x}) \\ \psi_{B\uparrow}(t, \vec{x}) \\ \psi_{A\downarrow}(t, \vec{x}) \\ \psi_{B\downarrow}(t, \vec{x}) \end{pmatrix} \quad (273)$$

where A, B label chiral (brane) degrees of freedom and \uparrow, \downarrow label two spin degrees of freedom that denote the two possible winding directions along a given direction \vec{e} .

In continuum limit, we have

$$\begin{aligned} \mathcal{H}_{\text{coupling}} &= \frac{J}{2} \sum_{\langle i, j \rangle} \psi_{A,i}^\dagger \psi_{B,j} + h.c. \\ &= \frac{J}{2} \sum_i \psi_i^\dagger (\hat{T}(a) \cdot \hat{S}) \psi_i + h.c. \\ &= \frac{J}{2} \sum_k \psi_k^\dagger (e^{ia(\vec{k} \cdot \vec{\sigma})} \cdot \hat{S}) \psi_k + h.c. \\ &= J \sum_k \psi_k^\dagger [\tau_y \otimes \sin(a(\vec{k} \cdot \vec{\sigma}))] \psi_k \end{aligned} \quad (274)$$

where \vec{k} is the wave vector along the direction \vec{e} , $\tau_y = i\hat{S} - iS^\dagger$, $c = aJ$ is the velocity. From above equation, we have the dispersion of knots,

$$E_k = J\tau_y \otimes \sin(ak) \quad (275)$$

where k is the value of the wave vector along the direction \vec{e} . There are two nodal points at $E_k = 0$, i.e., $k = 0$ and $k = \frac{\pi}{a}$. This dispersion is physical consequence of "two-sublattice" lattice. After considering an additional global gauge transformation $|\vec{k}\rangle' \rightarrow |\vec{k}\rangle = e^{i\Delta\phi} |\vec{k}\rangle$, we recover the generalized translation symmetry. In summary, we have

$$\text{A two-sublattice lattice model} \quad (276)$$

of Dirac knot-crystal

\implies Dirac model with linear dispersion.

Next, we consider the energy cost from rotating Dirac knot-crystal. For a balancedly-rotating Dirac knot-crystal, the rotating angular velocity is given by ω . Thus, for a balancedly-rotating Dirac knot-crystal, we have

$$\text{sgn}[\Upsilon(t)] = -\text{sgn}[\Upsilon(t + \frac{\pi}{\omega})]. \quad (277)$$

The rotating of Dirac knot-crystal leads to periodic varied knot states, i.e. at $t = 0$ we have $|i, A\rangle$; at $t = \frac{\pi}{\omega}$ we have $|i, B\rangle$. As a result, we use the following formulation to characterize the rotating brane,

$$\psi_i \rightarrow \psi'_i(t) = e^{\tau_x(i\omega t)} \cdot \psi_i. \quad (278)$$

After considering the energy from rotating of Dirac knot-crystal, a corresponding term is given by

$$\begin{aligned} i \sum_i \psi_i^\dagger \partial_t \psi'_i &= i \sum_i (\psi_i^\dagger e^{\tau_x(-i\omega t)}) \partial_t (e^{\tau_x(i\omega t)} \psi_i) \\ &= i \sum_i (\psi_i^\dagger e^{\tau_x(-i\omega t)}) \\ &\quad \cdot (e^{\tau_x(i\omega t)} \partial_t \psi_i + i\omega \tau_x \psi_i e^{\tau_x(i\omega t)}) \\ &= i \sum_i (\psi_i^\dagger \partial_t \psi_i - \psi_i^\dagger \omega \tau_x \psi_i). \end{aligned} \quad (279)$$

We write down the total Lagrangian of a 3D 1-level balancedly-rotating Dirac knot-crystal in continuum limit as

$$\begin{aligned} \mathcal{L} &= i \sum_i \psi_i^\dagger \partial_t \psi'_i - \mathcal{H}_{\text{coupling}} \\ &= i \int \psi^\dagger \partial_t \psi d^3x - \mathcal{H}_{3D} \end{aligned} \quad (280)$$

where

$$\begin{aligned} \mathcal{H}_{3D} &= \mathcal{H}_{\text{coupling}} + m \int \psi^\dagger \tau_x \psi d^3x \\ &= c \sum_k \psi_k^\dagger [\tau_y \otimes \sin(a(\vec{k} \cdot \vec{\sigma}))] \psi_k \\ &\quad + m \sum_k \psi_k^\dagger \tau_x \psi_k \end{aligned} \quad (281)$$

where $mc^2 = \hbar\omega$ plays role of the mass of knots. The term $\psi^\dagger[(\tau_y \otimes \vec{\sigma}) \cdot \hat{k}] \psi$ denotes the process of brane-curving along \vec{e} direction; the term $\psi^\dagger \tau_x \psi$ denotes the process of rotating of knot-crystal along time-axis.

From above equation, in the limit $k \rightarrow 0$ we have low energy effective Hamiltonian as

$$\begin{aligned} \mathcal{H}_{3D} &\simeq aJ \sum_k \psi_k^\dagger [\tau_y \otimes ((\vec{k} \cdot \vec{\sigma}))] \psi_k \\ &\quad + m \sum_k \psi_k^\dagger \tau_x \psi_k \\ &= c \int \psi^\dagger [(\tau_y \otimes \vec{\sigma}) \cdot \hat{k}] \psi d^3x \\ &\quad + m \int \psi^\dagger \tau_x \psi d^3x. \end{aligned} \quad (282)$$

We then re-write the effective Hamiltonian to be

$$\mathcal{H}_{3D} = \int (\psi^\dagger \hat{H}_{3D} \psi) d^3x \quad (283)$$

and

$$\hat{H}_{3D} = \vec{\Gamma} \cdot \vec{p} + m\Gamma^5$$

where $\Gamma^5 = \tau^x \otimes \mathbf{1}$, $\Gamma^x = \tau^y \otimes \sigma^x$, $\Gamma^y = \tau^y \otimes \sigma^y$, $\Gamma^z = \tau^y \otimes \sigma^z$, $\vec{p} = \hbar \vec{k}$ is the momentum operator. $\psi^\dagger = (\psi_{A,\uparrow,i}^\dagger, \psi_{B,\uparrow,i}^\dagger, \psi_{A,\downarrow,i}^\dagger, \psi_{B,\downarrow,i}^\dagger)$ is the annihilation operator of four-component fermions at the site i .

For the effective Hamiltonian \hat{H}_{3D} , there exists a global spacial rotation symmetry, $\hat{R}\hat{H}_{3D}\hat{R}^{-1} = \hat{H}_{3D}$ under the global spacial rotation operation $\hat{R} = \hat{R}_{\text{spin}} \cdot \hat{R}_{\text{space}}$ where \hat{R}_{spin} is SO(3) spin rotation operator $\hat{R}_{\text{spin}}\vec{\Gamma}\hat{R}_{\text{spin}}^{-1} = \vec{\Gamma}'$, and \hat{R}_{space} is SO(3) space rotation operator, $\hat{R}_{\text{space}}\vec{p}\hat{R}_{\text{space}}^{-1} = \vec{p}'$, $\hat{R}_{\text{space}}\vec{x}\hat{R}_{\text{space}}^{-1} = \vec{x}'$. After doing a global spacial rotation operation, the motion direction changes from \vec{e} to \vec{e}' .

Finally, the low energy effective Lagrangian of 3D 1-level uniform balancedly-rotating Dirac knot-crystal is simplified to be

$$\begin{aligned}\mathcal{L}_{3D} &= i\psi^\dagger \partial_t \psi - \mathcal{H}_{3D} \\ &= \bar{\psi}(i\gamma^\mu \hat{\partial}_\mu - m)\psi\end{aligned}\quad (284)$$

where $\bar{\psi} = \psi^\dagger \gamma^0$, γ^μ are the reduced Gamma matrices, $\gamma^1 = \gamma^0 \Gamma^1$, $\gamma^2 = \gamma^0 \Gamma^2$, $\gamma^3 = \gamma^0 \Gamma^3$, and $\gamma^0 = \Gamma^5 = \tau_x \otimes \mathbf{1}$, $\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$. This is Lagrangian for massive Dirac fermions with emergent SO(3,1) Lorentz-invariance. The SO(3,1) Lorentz transformations S_{Lor} is defined by $S_{\text{Lor}}\gamma^\mu S_{\text{Lor}}^{-1} = \gamma'^\mu$ ($\mu = 0, 1, 2, 3$) and $S_{\text{Lor}}\gamma^5 S_{\text{Lor}}^{-1} = \gamma^5$. Since the Fermi-velocity c only depends on the microscopic parameter J and a , we may regard c to be "light-velocity" and the invariance of light-velocity becomes an fundamental principle for the knot physics.

In summary, we have

$$\begin{aligned}\text{Balancedly-rotating velocity} \\ \implies \text{Mass of particle in Dirac model.}\end{aligned}\quad (285)$$

c. Excitations of 3D 1-level uniform balancedly-rotating knot-crystal The ground state of knot-crystal is a filled state of knots, of which the total number of knots is N_{total} . The excitation is a knot that is topological excitation.

For 3D balancedly-rotating knot-crystal, the energy dispersion of knots is

$$\begin{aligned}E_p &= \pm \sqrt{(J \sin(\pm ka))^2 + m^2 c^4} \\ &\approx \pm \sqrt{\hbar^2 k^2 c^2 + m^2 c^4}\end{aligned}\quad (286)$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$ and m is the mass gap. For a particle-like excitation, the energy is $\sqrt{\hbar^2 k^2 c^2 + m^2 c^4}$; for a hole-like excitation, the energy is $-\sqrt{\hbar^2 k^2 c^2 + m^2 c^4}$. For a particle-like excitation, an extra knot is put on the knot-crystal and the total number of a knots on 3-brane is $N_{\text{total}} + 1$; for a hole-like excitation, a knot is token off from the knot-crystal and the total number of a knots on 3-brane is $N_{\text{total}} - 1$. The situation is very similar to the quasi-particles in solid physics.

d. Schrödinger equation The deviation of the Schrödinger equation had been missed in last section. From above effective Lagrangian of balancedly-rotating knot-crystal, $\mathcal{L}_{3D} = \bar{\psi}(i\gamma^\mu \hat{\partial}_\mu - m)\psi$, we can easily obtain the Schrödinger equation in low velocity limit, $|\vec{p}| \rightarrow 0$,

$$\hat{H}\psi(\vec{x}, t) = -i\hbar \frac{d\psi(\vec{x}, t)}{dt}\quad (287)$$

where $\hat{H} = \frac{\vec{p}^2}{2m}$ and the Planck constant is defined by

$$\hbar = \frac{a \cdot k_B T_{\text{brane}}}{3\pi \cdot c_{5D}}$$

with c_{5D} is 5D light-speed and T_{brane} is the temperature of the branes T_{brane} that is equal to the $d + 2$ dimensional system T_{5D} .

D. Knot physics for 2-level double-helix knot-crystal

In particle physics, there exist three types fermionic elementary particles: neutrino, electron and quarks. To explain the existence of different types of elementary particles, people try to go beyond SM. Pati and Salam [20] had proposed preons to be the fundamental constituent particles. Then, other types of models were developed, for example, the Rishon Model proposed simultaneously by Harari and Shupe [21, 22], the helon model by S. O. Bilson-Thompson[23], the tangles by C. Schiller[6].

In this section we show that the effective theory for the 2-level double-helix knot-crystal is quantum gauge theory – an interacting quantum field theory, in which Dirac fermions couple gauge fields. After mapping 2-level double-helix knot-crystal to 2-level composite knot-crystal, the composite knots become elementary particles in particle physics, including neutrino, electron and quarks. The collective quantum fluctuations of the composite knot-crystal become gauge fields – position-fluctuations of internal-windings are Abelian gauge field and order-fluctuations of internal-windings are non-Abelian gauge field.

1. 2-level double-helix knot-crystal

For a 2-level double-helix knot-crystal, there also exist eight possible chiralities, i.e., $\mathbf{Z}_{L,LL}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{L,LR}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{L,RL}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{L,RR}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{R,LL}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{R,LR}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{R,RL}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{R,RR}(\vec{e}, \vec{x}, t)$.

We then show the knot-functions of 2-level double-helix knot-crystal for $\mathbf{Z}_{L,IJ}(\vec{e}, \vec{x}, t)$ ($I, J = L, R$)

$$\begin{aligned} \mathbf{Z}_{L,IJ}(\vec{e}, \vec{x}, t) &= \begin{pmatrix} z_{A,1}(\Delta\phi_{A,2}(\vec{x}), t) & z_{A,2}(\vec{x}) \\ z_{B,1}(\Delta\phi_{B,2}(\vec{x}), t) & z_{B,2}(\vec{x}) \end{pmatrix} \\ &= \begin{pmatrix} r_{A,1}e^{\pm i\Delta\phi_{A,1}(\Delta\phi_{A,2}(\vec{x})) + i\omega_A t} & r_{A,2}e^{i\Delta\phi_{A,2}(\vec{x})} \\ r_{B,1}e^{\pm i\Delta\phi_{B,1}(\Delta\phi_{B,2}(\vec{x})) + i\omega_B t} & r_{B,2}e^{i\Delta\phi_{B,2}(\vec{x})} \end{pmatrix} \end{aligned} \quad (288)$$

where

$$\Delta\phi_{A,2}(\vec{x}) = \Delta\phi_{B,2}(\vec{x}) = \frac{\pi}{a} \vec{e} \cdot \vec{x} \quad (289)$$

and

$$\begin{aligned} \Delta\phi_{A,1}(\Delta\phi_{A,2}(\vec{x})) &= (\Delta_{12})_A \Delta\phi_{A,2}(\vec{x}), \\ \Delta\phi_{B,1}(\Delta\phi_{B,2}(\vec{x})) &= (\Delta_{12})_B \Delta\phi_{B,2}(\vec{x}). \end{aligned} \quad (290)$$

a is a fixed length that denotes the half pitch of windings of level-2 knot-crystal. \vec{e} is winding direction. The period ratios $(\Delta_{12})_{A/B}$ are the (positive) winding numbers of A/B brane in level-2. $r_{A/B,1}$ and $r_{A/B,2}$ are winding radii for level-1 knot-crystal and level-2 knot-crystal, respectively. In general, we have $r_{A,2} = r_{B,2} = r_2$ and $r_{A,1} \simeq r_{B,1} = r_1$.

For $\mathbf{Z}_{R,IJ}(\vec{e}, \vec{x}, t)$ ($I, J = L, R$), the knot-function becomes

$$\begin{aligned} \mathbf{Z}_{R,IJ}(\vec{e}, \vec{x}, t) &= \begin{pmatrix} z_{A,1}(\Delta\phi_{A,2}(\vec{x}), t) & z_{A,2}(\vec{x}) \\ z_{B,1}(\Delta\phi_{B,2}(\vec{x}), t) & z_{B,2}(\vec{x}) \end{pmatrix} \\ &= \begin{pmatrix} r_{A,1}e^{\pm i\Delta\phi_{A,1}(\Delta\phi_{A,2}(\vec{x})) + i\omega_A t} & r_{A,2}e^{i\Delta\phi_{A,2}(\vec{x})} \\ r_{B,1}e^{\pm i\Delta\phi_{B,1}(\Delta\phi_{B,2}(\vec{x})) + i\omega_B t} & r_{B,2}e^{i\Delta\phi_{B,2}(\vec{x})} \end{pmatrix} \end{aligned} \quad (291)$$

where

$$\Delta\phi_{A,2}(\vec{x}, t) = \Delta\phi_{B,2}(\vec{x}, t) = -\frac{\pi}{a} \vec{e} \cdot \vec{x} \quad (292)$$

and

$$\begin{aligned} \Delta\phi_{A,1}(\Delta\phi_{A,2}(\vec{x})) &= (\Delta_{12})_A \Delta\phi_{A,2}(\vec{x}), \\ \Delta\phi_{B,1}(\Delta\phi_{B,2}(\vec{x})) &= (\Delta_{12})_B \Delta\phi_{B,2}(\vec{x}). \end{aligned} \quad (293)$$

2. 2-level composite knot-crystal with $(\Delta_{12})_{A/B} = n + \frac{1}{2}$

In this part we focus on a special 3D 2-level double-helix knot-crystal $\mathbf{Z}(\vec{e}, \vec{x}, t)$, of which we have

$$(\Delta_{12})_A = (\Delta_{12})_B = n + \frac{1}{2} \quad (294)$$

where n is an integer number. We take 3D 2-level knot-crystal with knot-function $\mathbf{Z}_{L,LL}(\vec{e}, \vec{x}, t)$ as an example to learn knot physics for 2-level double-helix knot-crystal.

a. Composite knot-crystal Firstly, we split a 2-level double-helix knot-crystal with $(\Delta_{12})_{A/B} = n + \frac{1}{2}$ to two 2-level double-helix knot-crystals – one is a 2-level knot-crystal with $(\Delta_{12})_{A/B} = \frac{1}{2}$ and the other is 2-level double-helix knot-crystal with $(\Delta_{12})_{A/B} = n$, i.e.,

$$A \ (\Delta_{12})_{A/B} = n + \frac{1}{2} \text{ 2-level knot-crystal} \quad (295)$$

$$\implies A \ (\Delta_{12})_{A/B} = \frac{1}{2} \text{ 2-level knot-crystal} \quad (296)$$

$$+ A \ (\Delta_{12})_{A/B} = n \text{ 2-level knot-crystal.}$$

Such splitting can be characterized by using composite representation of a 2-level double-helix knot-crystal as

$$\mathbf{Z}_{\text{L,LL}}(\vec{e}, \vec{x}, t) = \mathbf{Z}_{\text{Dirac}}(\vec{x}, t) \otimes \mathbf{Z}_{\text{substrate}}(\vec{x}) \quad (297)$$

where $\mathbf{Z}_{\text{Dirac}}(\vec{x}, t)$ is the knot-function of a 2-level *Dirac* knot-crystal with $(\Delta_{12})_{\text{A/B}} = \frac{1}{2}$ as

$$\begin{aligned} \mathbf{Z}_{\text{Dirac}}(\vec{x}) &= \begin{pmatrix} z_{\text{Dirac,A,1}}(\Delta\phi_{\text{A,2}}(\vec{x})) & z_{\text{Dirac,A,2}}(\vec{x}) \\ z_{\text{Dirac,B,1}}(\Delta\phi_{\text{B,2}}(\vec{x})) & z_{\text{Dirac,B,2}}(\vec{x}) \end{pmatrix} \\ &= \begin{pmatrix} r_{\text{A,1}}e^{i\frac{\pi}{2a}\vec{e}\cdot\vec{x}} & r_{\text{A,2}}e^{i\frac{\pi}{a}\vec{e}\cdot\vec{x}} \\ r_{\text{B,1}}e^{i\frac{\pi}{2a}\vec{e}\cdot\vec{x}} & r_{\text{B,2}}e^{i\frac{\pi}{a}\vec{e}\cdot\vec{x}} \end{pmatrix} \end{aligned} \quad (298)$$

and $\mathbf{Z}_{\text{substrate}}(\vec{x})$ is the knot-function of a 2-level knot-crystal with $(\Delta_{12})_{\text{A/B}} = n$ as

$$\begin{aligned} \mathbf{Z}_{\text{substrate}}(\vec{x}) &= \begin{pmatrix} z_{\text{substrate,A,1}}(\Delta\phi_{\text{A,2}}(\vec{x})) & z_{\text{substrate,A,2}}(\vec{x}) \\ z_{\text{substrate,B,1}}(\Delta\phi_{\text{B,2}}(\vec{x})) & z_{\text{substrate,B,2}}(\vec{x}) \end{pmatrix} \\ &= \begin{pmatrix} r_{\text{A,1}}e^{i\frac{\pi}{a}\vec{e}\cdot\vec{x}n} & r_{\text{A,2}}e^{i\frac{\pi}{a}\vec{e}\cdot\vec{x}} \\ r_{\text{B,1}}e^{i\frac{\pi}{a}\vec{e}\cdot\vec{x}n} & r_{\text{B,2}}e^{i\frac{\pi}{a}\vec{e}\cdot\vec{x}} \end{pmatrix}. \end{aligned} \quad (299)$$

In the following part, we call the 2-level knot-crystal with $(\Delta_{12})_{\text{A/B}} = n$ described by $\mathbf{Z}_{\text{substrate}}(\vec{x}, t)$ to be a *substrate* knot-crystal.

As a result, a 2-level double-helix knot-crystal with $(\Delta_{12})_{\text{A/B}} = n + \frac{1}{2}$ can be regarded as a composite knot-crystal by combing two knot-crystals – one is a 2-level knot-crystal with $(\Delta_{12})_{\text{A/B}} = \frac{1}{2}$ and the other with $(\Delta_{12})_{\text{A/B}} = n$. We may use the knot-crystal-operation $\hat{U}(\mathbf{Z}_{\text{L,LL}})$, $\hat{U}(\mathbf{Z}_{\text{Dirac}})$, $\hat{U}(\mathbf{Z}_{\text{substrate}})$ to show composite knot-crystal.

Firstly, we define $\hat{U}(\mathbf{Z}_{\text{L,LL}})$ as

$$\begin{aligned} \hat{U}(\mathbf{Z}_{\text{L,LL}}) &= \hat{L}[\hat{U}(z_{\text{A,1}}(\vec{x}, t))\hat{U}(z_{\text{B,1}}(\vec{x})) \\ &\quad \hat{U}(z_{\text{A,2}}(\vec{x}, t))\hat{U}(z_{\text{B,2}}(\vec{x}))], \end{aligned} \quad (300)$$

where

$$\begin{aligned} \hat{U}(z_{\text{A,2}}(\vec{x})) &= \exp[i\Delta\phi_{\text{A,2}}(\vec{x}) \cdot \hat{K}], \\ \hat{U}(z_{\text{B,2}}(\vec{x})) &= \exp[i\Delta\phi_{\text{B,2}}(\vec{x}) \cdot \hat{K}] \end{aligned} \quad (301)$$

and

$$\begin{aligned} \hat{U}(z_{\text{A,1}}(\vec{x})) &= \exp[i\Delta\phi_{\text{A,1}}(\Delta\phi_{\text{A,2}}(\vec{x})) \cdot \hat{K}], \\ \hat{U}(z_{\text{B,1}}(\vec{x})) &= \exp[i\Delta\phi_{\text{B,1}}(\Delta\phi_{\text{B,2}}(\vec{x})) \cdot \hat{K}] \end{aligned} \quad (302)$$

where $\hat{K} = -i\frac{d}{d\phi_{\text{A/B,1/2}}}$, $\Delta\phi_{\text{A/B,1}}(\Delta\phi_{\text{A/B,2}}(\vec{x})) = (\Delta_{12})_{\text{A/B}} \Delta\phi_{\text{A/B,2}}(\vec{x}) = (n + \frac{1}{2}) \cdot \Delta\phi_{\text{A/B,2}}(\vec{x})$, and $\Delta\phi_{\text{A/B,2}}(\vec{x}, t) = \vec{k}_0 \cdot \vec{x}$. $\vec{k}_0 = \frac{\pi}{a}\vec{e}$ denotes winding vector along the winding direction, and a is the length that denotes the half pitch of largest windings of i -th brane. \pm denotes the winding chirality.

Secondly, we define $\hat{U}(\mathbf{Z}_{\text{Dirac}})$

$$\begin{aligned} \hat{U}(\mathbf{Z}_{\text{Dirac}}) &= \hat{L}[\hat{U}(z_{\text{A,1}}(\vec{x}, t))\hat{U}(z_{\text{B,1}}(\vec{x})) \\ &\quad \hat{U}(z_{\text{A,2}}(\vec{x}, t))\hat{U}(z_{\text{B,2}}(\vec{x}))], \end{aligned} \quad (303)$$

where

$$\begin{aligned} \hat{U}(\mathbf{Z}_{\text{Dirac,A,2}}(\vec{x})) &= \exp[i\Delta\phi_{\text{A,2}}(\vec{x}) \cdot \hat{K}], \\ \hat{U}(\mathbf{Z}_{\text{Dirac,B,2}}(\vec{x})) &= \exp[i\Delta\phi_{\text{B,2}}(\vec{x}) \cdot \hat{K}] \end{aligned} \quad (304)$$

and

$$\begin{aligned} \hat{U}(\mathbf{Z}_{\text{Dirac,A,1}}(\Delta\phi_{\text{A,2}}(\vec{x}))) &= \exp[i\Delta\phi_{\text{A,1}}(\Delta\phi_{\text{A,2}}(\vec{x})) \cdot \hat{K}], \\ \hat{U}(\mathbf{Z}_{\text{Dirac,B,1}}(\Delta\phi_{\text{B,2}}(\vec{x}))) &= \exp[i\Delta\phi_{\text{B,1}}(\Delta\phi_{\text{B,2}}(\vec{x})) \cdot \hat{K}] \end{aligned} \quad (305)$$

where $\hat{K} = -i \frac{d}{d\phi_{A/B,1/2}}$, $\Delta\phi_{A/B,1}(\Delta\phi_{A/B,2}(\vec{x})) = (\Delta_{12})_{A/B} \Delta\phi_{A/B,2}(\vec{x}) = \frac{1}{2} \cdot \Delta\phi_{A/B,2}$, and $\Delta\phi_{A/B,2}(\vec{x}, t) = \vec{k}_0 \cdot \vec{x}$.
 Thirdly, we define $\hat{U}(\mathbf{Z}_{\text{substrate}})$

$$\begin{aligned} \hat{U}(\mathbf{Z}_{\text{substrate}}) &= \hat{L}[\hat{U}(z_{A,1}(\vec{x}, t))\hat{U}(z_{B,1}(\vec{x})) \\ &\quad \hat{U}(z_{A,2}(\vec{x}, t))\hat{U}(z_{B,2}(\vec{x}))], \end{aligned} \quad (306)$$

where

$$\begin{aligned} \hat{U}(\mathbf{Z}_{\text{substrate},A,2}(\vec{x})) &= \exp[i\Delta\phi_{A,2}(\vec{x}) \cdot \hat{K}], \\ \hat{U}(\mathbf{Z}_{\text{substrate},B,2}(\vec{x})) &= \exp[i\Delta\phi_{B,2}(\vec{x}) \cdot \hat{K}] \end{aligned} \quad (307)$$

and

$$\begin{aligned} \hat{U}(\mathbf{Z}_{\text{substrate},A,1}(\Delta\phi_{A,2}(\vec{x}))) &= \exp[i\Delta\phi_{A,1}(\Delta\phi_{A,2}(\vec{x})) \cdot \hat{K}], \\ \hat{U}(\mathbf{Z}_{\text{substrate},B,1}(\Delta\phi_{B,2}(\vec{x}))) &= \exp[i\Delta\phi_{B,1}(\Delta\phi_{B,2}(\vec{x})) \cdot \hat{K}] \end{aligned} \quad (308)$$

where $\hat{K} = -i \frac{d}{d\phi_{A/B,1/2}}$, $\Delta\phi_{A/B,1}(\Delta\phi_{A/B,2}(\vec{x})) = (\Delta_{12})_{A/B} \Delta\phi_{A/B,2}(\vec{x}) = n \cdot \Delta\phi_{A/B,2}$, and $\Delta\phi_{A/B,2}(\vec{x}, t) = \vec{k}_0 \cdot \vec{x}$.
 The knot-functions can be obtained by corresponding knot-crystal-operation on flat branes as

$$\begin{aligned} \mathbf{Z}_{L,LL}(\vec{e}, \vec{x}, t) &= \hat{U}(\mathbf{Z}_{L,LL}) \begin{pmatrix} r_{A,1} & r_{A,2} \\ r_{B,1} & r_{B,2} \end{pmatrix}, \\ \mathbf{Z}_{\text{Dirac}}(\vec{e}, \vec{x}, t) &= \hat{U}(\mathbf{Z}_{\text{Dirac}}) \begin{pmatrix} r_{A,1} & r_{A,2} \\ r_{B,1} & r_{B,2} \end{pmatrix}, \\ \mathbf{Z}_{\text{substrate}}(\vec{e}, \vec{x}, t) &= \hat{U}(\mathbf{Z}_{\text{substrate}}) \begin{pmatrix} r_{A,1} & r_{A,2} \\ r_{B,1} & r_{B,2} \end{pmatrix}. \end{aligned} \quad (309)$$

Thus, the combination of composite knot-crystal $\mathbf{Z}(\vec{e}, \vec{x}, t)$ by $\mathbf{Z}_{\text{Dirac}}(\vec{e}, \vec{x}, t)$ and $\mathbf{Z}_{\text{substrate}}(\vec{e}, \vec{x}, t)$ is represented by

$$\begin{aligned} \hat{U}(z_{A,2}(\vec{x})) &= \hat{U}(\mathbf{Z}_{\text{Dirac},A,2}(\vec{x})) = \hat{U}(\mathbf{Z}_{\text{substrate},A,2}(\vec{x})), \\ \hat{U}(z_{B,2}(\vec{x})) &= \hat{U}(\mathbf{Z}_{\text{Dirac},B,2}(\vec{x})) = \hat{U}(\mathbf{Z}_{\text{substrate},B,2}(\vec{x})), \end{aligned} \quad (310)$$

and

$$\begin{aligned} \hat{U}(z_{A,1}(\Delta\phi_{A,2}(\vec{x}))) &= \hat{U}(\mathbf{Z}_{\text{Dirac},A,1}(\Delta\phi_{A,2}(\vec{x}))) \\ &\quad \cdot \hat{U}(\mathbf{Z}_{\text{substrate},A,1}(\Delta\phi_{A,2}(\vec{x}))), \\ \hat{U}(z_{B,1}(\Delta\phi_{B,2}(\vec{x}))) &= \hat{U}(\mathbf{Z}_{\text{Dirac},B,1}(\Delta\phi_{B,2}(\vec{x}))) \\ &\quad \cdot \hat{U}(\mathbf{Z}_{\text{substrate},B,1}(\Delta\phi_{B,2}(\vec{x}))). \end{aligned} \quad (311)$$

b. Generalized translation symmetry for composite knot-crystal In this part we discuss the generalized spacial translation symmetry for composite knot-crystal.

Generating the composite knot-crystal by knot-crystal-operation Before discussing the translation symmetry, we generate the composite knot-crystal $\mathbf{Z}(\vec{e}, \vec{x}, t) = \mathbf{Z}_{\text{Dirac}}(\vec{x}, t) \circledast \mathbf{Z}_{\text{substrate}}(\vec{x})$ step-by-step.

At first step, we generate the composite knot-crystal $\mathbf{Z}_{\text{substrate}}(\vec{x})$ by knot-crystal-operation $\hat{U}(\mathbf{Z}_{\text{substrate}})$ that is

$$\begin{aligned} \hat{U}(\mathbf{Z}_{\text{substrate}}) &= \hat{L}[\hat{U}(z_{A,1}(\vec{x}, t))\hat{U}(z_{B,1}(\vec{x})) \\ &\quad \hat{U}(z_{A,2}(\vec{x}, t))\hat{U}(z_{B,2}(\vec{x}))]. \end{aligned} \quad (312)$$

Next, we do knot-crystal-operation $\hat{U}(\mathbf{Z}_{\text{Dirac},A/B,1}(\Delta\phi_{A/B,2}(\vec{x})))$ on composite knot-crystal $\mathbf{Z}_{\text{substrate}}(\vec{x})$ that is

$$\begin{aligned} \hat{U}(\mathbf{Z}_{\text{Dirac},A,1}(\Delta\phi_{A,2}(\vec{x}))) &= \exp[i\Delta\phi_{A,1}(\Delta\phi_{A,2}(\vec{x})) \cdot \hat{K}], \\ \hat{U}(\mathbf{Z}_{\text{Dirac},B,1}(\Delta\phi_{B,2}(\vec{x}))) &= \exp[i\Delta\phi_{B,1}(\Delta\phi_{B,2}(\vec{x})) \cdot \hat{K}]. \end{aligned} \quad (313)$$

Finally, we derive the composite knot-crystal $\mathbf{Z}(\vec{x})$ as

$$\begin{aligned}\mathbf{Z}(\vec{e}, \vec{x}, t) &= \hat{U}(\mathbf{Z}_{\text{Dirac}, A, 1}(\Delta\phi_{A, 2}(\vec{x}))) \cdot \hat{U}(\mathbf{Z}_{\text{Dirac}, B, 1}(\Delta\phi_{B, 2}(\vec{x}))) \\ &\quad \cdot \hat{U}(\mathbf{Z}_{\text{substrate}}) \begin{pmatrix} r_{A, 1} & r_{A, 2} \\ r_{B, 1} & r_{B, 2} \end{pmatrix} \\ &= \hat{U}(\mathbf{Z}_{\text{Dirac}, A, 1}(\Delta\phi_{A, 2}(\vec{x}))) \cdot \hat{U}(\mathbf{Z}_{\text{Dirac}, B, 1}(\Delta\phi_{B, 2}(\vec{x}))) \\ &\quad \cdot \mathbf{Z}_{\text{substrate}}(\vec{e}, \vec{x}, t).\end{aligned}\tag{314}$$

Global translation symmetry We define the global translation operation for the composite knot-crystal

$$\hat{\mathcal{T}}_{\text{global}}(\Delta\vec{x}) = \hat{\mathcal{T}}_{\text{Dirac}}(\Delta\vec{x})\hat{\mathcal{T}}_{\text{substrate}}(\Delta\vec{x})\tag{315}$$

where $\hat{\mathcal{T}}_{\text{Dirac}}(\Delta\vec{x})$ is the translation operation on 2-level Dirac knot-crystal with $(\Delta_{12})_{A/B} = \frac{1}{2}$ described by $\mathbf{Z}_{\text{Dirac}}(\vec{x}, t)$ and $\hat{\mathcal{T}}_{\text{substrate}}(\Delta\vec{x})$ is the translation operation on 2-level substrate knot-crystal described by $\mathbf{Z}_{\text{substrate}}(\vec{x}, t)$, respectively.

Then, we show the global translation symmetry of the composite knot-crystal,

$$\begin{aligned}\mathbf{Z}(\vec{e}, \vec{x}, t) &\rightarrow \hat{\mathcal{T}}_{\text{global}}(\Delta\vec{x})\mathbf{Z}(\vec{e}, \vec{x}, t) = [\hat{\mathcal{T}}_{\text{Dirac}}(\Delta\vec{x})\mathbf{Z}_{\text{Dirac}}(\vec{x}, t)] \otimes [\hat{\mathcal{T}}_{\text{substrate}}(\Delta\vec{x})\mathbf{Z}_{\text{substrate}}(\vec{x})] \\ &= \mathbf{Z}_{\text{Dirac}}(\vec{x} + \Delta\vec{x}, t) \otimes \mathbf{Z}_{\text{substrate}}(\vec{x} + \Delta\vec{x}) \\ &= \mathbf{Z}(\vec{e}, \vec{x} + \Delta\vec{x}, t).\end{aligned}\tag{316}$$

For $\mathbf{Z}_{L, IJ}(\vec{e}, \vec{x}, t)$ ($I, J = L, R$), we have

$$\begin{aligned}\mathcal{T}_{\text{global}}(\Delta\vec{x}) \begin{pmatrix} z_{A, 1}(\Delta\phi_{A, 2}(\vec{x}), t) \\ z_{B, 1}(\Delta\phi_{B, 2}(\vec{x}), t) \end{pmatrix} &= \begin{pmatrix} z_{A, 1}(\Delta\phi_{A, 2}(\vec{x} + \Delta\vec{x}), t) \\ z_{B, 1}(\Delta\phi_{B, 2}(\vec{x} + \Delta\vec{x}), t) \end{pmatrix} \\ &= \begin{pmatrix} \exp[\pm i \frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x})(\Delta_{12})_A] z_{A, 1}(\Delta\phi_{A, 2}(\vec{x}), t) \\ \exp[\pm i \frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x})(\Delta_{12})_B] z_{B, 1}(\Delta\phi_{B, 2}(\vec{x}), t) \end{pmatrix}; \\ \mathcal{T}_{\text{global}}(\Delta\vec{x}) \begin{pmatrix} z_{A, 2}(\vec{x}) \\ z_{B, 2}(\vec{x}) \end{pmatrix} &= \begin{pmatrix} z_{A, 2}(\vec{x} + \Delta\vec{x}) \\ z_{B, 2}(\vec{x} + \Delta\vec{x}) \end{pmatrix} \\ &= \begin{pmatrix} \exp[i \frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x})] z_{A, 2}(\vec{x}) \\ \exp[i \frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x})] z_{B, 2}(\vec{x}) \end{pmatrix};\end{aligned}\tag{317}$$

for $\mathbf{Z}_{R, IJ}(\vec{e}, \vec{x}, t)$ ($I, J = L, R$), we have

$$\begin{aligned}\mathcal{T}_{\text{global}}(\Delta\vec{x}) \begin{pmatrix} z_{A, 1}(\Delta\phi_{A, 2}(\vec{x}), t) \\ z_{B, 1}(\Delta\phi_{B, 2}(\vec{x}), t) \end{pmatrix} &= \begin{pmatrix} z_{A, 1}(\Delta\phi_{A, 2}(\vec{x} + \Delta\vec{x}), t) \\ z_{B, 1}(\Delta\phi_{B, 2}(\vec{x} + \Delta\vec{x}), t) \end{pmatrix} \\ &= \begin{pmatrix} \exp[\pm i \frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x})(\Delta_{12})_A] z_{A, 1}(\Delta\phi_{A, 2}(\vec{x}), t) \\ \exp[\pm i \frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x})(\Delta_{12})_B] z_{B, 1}(\Delta\phi_{B, 2}(\vec{x}), t) \end{pmatrix}, \\ \mathcal{T}(\Delta\vec{x}) \begin{pmatrix} z_{A, 2}(\vec{x}) \\ z_{B, 2}(\vec{x}) \end{pmatrix} &= \begin{pmatrix} \mathcal{T}(\Delta\vec{x}) z_{A, 2}(\vec{x} + \Delta\vec{x}) \\ \mathcal{T}(\Delta\vec{x}) z_{B, 2}(\vec{x} + \Delta\vec{x}) \end{pmatrix} \\ &= \begin{pmatrix} \exp[-i \frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x})] z_{A, 2}(\vec{x}) \\ \exp[-i \frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x})] z_{B, 2}(\vec{x}) \end{pmatrix}.\end{aligned}\tag{318}$$

In physics, we cannot distinguish $\mathbf{Z}(\vec{e}, \vec{x}, t)$ and $\hat{\mathcal{T}}_{\text{global}}(\Delta\vec{x})\mathbf{Z}(\vec{e}, \vec{x}, t)$. Thus, we call this translation symmetry for $\mathbf{Z}(\vec{e}, \vec{x}, t)$ to be *global translation symmetry*. In addition, according to the generalized translation symmetry, both the composite knot and its n -internal-windings have uniform distribution of $\mathbf{Z}_{\text{Dirac}}(\vec{e}, \vec{x}, t)$ and $\mathbf{Z}_{\text{substrate}}(\vec{e}, \vec{x}, t)$.

c. Internal translation symmetry We define the internal translation operation for the composite knot-crystal

$$\hat{\mathcal{T}}_{\text{internal}}(\Delta\vec{x}) = \hat{\mathcal{T}}_{\text{Dirac}}(\frac{\Delta\vec{x}}{2})\hat{\mathcal{T}}_{\text{substrate}}(-\frac{\Delta\vec{x}}{2}).$$

We show the internal translation symmetry of the composite knot-crystal, $\mathbf{Z}(\vec{e}, \vec{x}, t)$, i.e.,

$$\begin{aligned}\mathbf{Z}(\vec{e}, \vec{x}, t) &\rightarrow \hat{\mathcal{T}}_{\text{internal}}(\Delta\vec{x})\mathbf{Z}(\vec{e}, \vec{x}, t) = [\hat{\mathcal{T}}_{\text{Dirac}}(\frac{\Delta\vec{x}}{2})\mathbf{Z}_{\text{Dirac}}(\vec{x}, t)] \\ &\quad \otimes [\hat{\mathcal{T}}_{\text{substrate}}(-\frac{\Delta\vec{x}}{2})\mathbf{Z}_{\text{substrate}}(\vec{x})] \\ &= \mathbf{Z}_{\text{Dirac}}(\vec{x} + \frac{\Delta\vec{x}}{2}, t) \otimes \mathbf{Z}_{\text{substrate}}(\vec{x} - \frac{\Delta\vec{x}}{2}).\end{aligned}\tag{319}$$

In physics, we also cannot distinguish $\mathbf{Z}(\vec{e}, \vec{x}, t) = \mathbf{Z}_{\text{Dirac}}(\vec{x}, t) \otimes \mathbf{Z}_{\text{substrate}}(\vec{x})$ and $\hat{T}_{\text{internal}}(\Delta\vec{x})\mathbf{Z}(\vec{e}, \vec{x}, t) = \mathbf{Z}_{\text{Dirac}}(\vec{x} + \frac{\Delta\vec{x}}{2}, t) \otimes \mathbf{Z}_{\text{substrate}}(\vec{x} - \frac{\Delta\vec{x}}{2})$. Thus, we call this translation symmetry to be *internal translation symmetry*.

d. Composite knots The knots of 2-level composite knot-crystal are a 2-th level winding with a half 1-th winding and n_k -internal-winding. We call such composite object (a 2-th level winding with a half 1-th winding and n_k -internal-winding) to be a *linear composite knot*, i.e.,

$$\begin{aligned} & \text{Composite knot} \\ &= \text{A 2-th level winding} \end{aligned} \quad (320)$$

$$\begin{aligned} &+ \text{a bare knot } \left(\frac{1}{2} \text{ 1-th winding}\right) \\ &+ n_k\text{-internal-winding.} \end{aligned} \quad (321)$$

A knot of 2-level double-helix knot-crystal with $(\Delta_{12})_{A/B} = \frac{1}{2}$ is called to be *bare knot* and a 1-th-winding of 2-level double-helix knot-crystal with $(\Delta_{12})_{A/B} = n_k$ is called to be *internal winding* ($n_k = 0, 1, 2, \dots, n$). There are $n + 1$ types of four-component (brane A/B and spin up/down) linear composite knot on double-helix knot-crystal, i.e., $n_k = 0, 1, 2, \dots, n$.

A linear composite knot is a 2-th level winding along a given direction \vec{e} , of which the knot-function is defined as

$$\begin{aligned} \mathbf{Z}_{\text{knot},A}(\vec{x}, t) &= \begin{pmatrix} z_{A,1}(\Delta\phi_{A,2}(\vec{x}), t) & z_{A,2}(\vec{x}) \\ z_{B,1}(\Delta\phi_{B,2}(\vec{x}), t) & z_{B,2}(\vec{x}) \end{pmatrix} \\ &= \begin{pmatrix} r_{A,1} \exp[\pm i\phi_{\text{knot},1}(\Delta\phi_{A,2}(\vec{x}), t)] & r_{A,2} \exp[\pm i\phi_{\text{knot},2}(x, t)] \\ r_{B,1} & r_{B,2} \end{pmatrix} \end{aligned} \quad (322)$$

and

$$\mathbf{Z}_{\text{knot},B}(\vec{x}, t) = \begin{pmatrix} z_{A,1}(\Delta\phi_{A,2}(\vec{x}), t) & z_{A,2}(\vec{x}) \\ z_{B,1}(\Delta\phi_{B,2}(\vec{x}), t) & z_{B,2}(\vec{x}) \end{pmatrix} \quad (323)$$

$$= \begin{pmatrix} r_{A,1} & r_{A,2} \\ r_{B,1} \exp[\pm i\phi_{\text{knot},1}(\Delta\phi_{B,2}(\vec{x}), t)] & r_{B,2} \exp[\pm i\phi_{\text{knot},2}(\vec{x})] \end{pmatrix} \quad (324)$$

where

$$\phi_{\text{knot},2}(x'_1) = \begin{cases} 0, & x'_1 \in (-\infty, x_0] \\ -k_0(x'_1 - x_0), & x'_1 \in (x_0, x_0 + 2a] \\ 2\pi, & x'_1 \in (x_0 + 2a, \infty) \end{cases}, \quad (325)$$

and

$$\phi_{\text{knot},1}(\Delta\phi_{A/B,2}) = \begin{cases} \phi_0 - \frac{\pi}{2}, & \Delta\phi_{A/B,2} \in (-\infty, -\pi] \\ \phi_0 - \frac{\pi}{2} - \frac{(2n_k+1)}{2}\Delta\phi_{A/B,2}, & \Delta\phi_{A/B,2} \in (0, 2\pi] \\ \phi_0 + \frac{\pi}{2}, & \Delta\phi_{A/B,2} \in (\pi, \infty) \end{cases}. \quad (326)$$

$\Delta\phi_{A/B,1}(\Delta\phi_{A/B,2}(\vec{x})) = (\Delta_{12})_{A/B} \Delta\phi_{A/B,2}(\vec{x}) = n \cdot \Delta\phi_{A/B,2}$, and $\Delta\phi_{A/B,2}(\vec{x}, t) = \vec{k}_0 \cdot \vec{x}$. $x'_1 = \vec{x} \cdot \vec{e}$ is the coordination on the axis along a given direction \vec{e} and ϕ_0 is a constant angle, $k_0 = \frac{\pi}{a}$.

We may also generate a linear composite knot by doing knot-operation on composite knot-crystal

$$\begin{aligned} \mathbf{Z}_{\text{knot},A/B}(\vec{x}, t) &= \begin{pmatrix} z_{A,1}(\Delta\phi_{A,2}(\vec{x})) & z_{A,2}(\vec{x}) \\ z_{B,1}(\Delta\phi_{B,2}(\vec{x})) & z_{B,2}(\vec{x}) \end{pmatrix} \\ &= \mathbf{U}_{\text{knot}} \mathbf{Z}_{L,LL}(\vec{x}, t) \\ &= \mathbf{U}_{\text{knot},1} \mathbf{U}_{\text{knot},2} \mathbf{Z}_{L,LL}(\vec{x}, t) \end{aligned} \quad (327)$$

where

$$\begin{aligned} \mathbf{U}_{\text{knot},2} &= \exp[\pm i\phi_{\text{knot},A/B,2}(x'_1) \cdot \hat{K}], \\ \phi_{\text{knot},A/B,2}(x'_1) &= \begin{cases} 0, & x'_1 \in (-\infty, x_0] \\ -k_0(x'_1 - x_0), & x'_1 \in (x_0, x_0 + 2a] \\ 2\pi, & x'_1 \in (x_0 + 2a, \infty) \end{cases}, \end{aligned} \quad (328)$$

and

$$\mathbf{U}_{\text{knot},1} = \exp[i\phi_{\text{knot},A/B,1}(\Delta\phi_{A/B,2}) \cdot \hat{K}], \quad (329)$$

with

$$\phi_{\text{knot}}(\Delta\phi_{A/B,2}) = \left\{ \begin{array}{ll} \phi_0 - \frac{\pi}{2}, & \Delta\phi_{A/B,2} \in (-\infty, -\pi] \\ \phi_0 - \frac{\pi}{2} - \frac{(2n_k+1)}{2}\Delta\phi_{A/B,2}, & \Delta\phi_{A/B,2} \in (0, 2\pi] \\ \phi_0 + \frac{\pi}{2}, & \Delta\phi_{A/B,2} \in (\pi, \infty) \end{array} \right\} \quad (330)$$

where $\hat{K} = -i\frac{d}{d\phi_{A/B,1/2}}$ is knot-number operator.

3. Quantum states of composite knot-crystal

a. Quantum states of composite knot-crystal For a 2-level composite knot-crystal with $(\Delta_{12})_{A/B} = n + \frac{1}{2}$, the quantum state for a composite knot is denoted by

$$\begin{aligned} & \left| \vec{j} \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle \\ &= \left| \vec{j} \cdot a, \tau, \sigma \right\rangle \otimes (|\delta_1\rangle_{\vec{j}} \otimes |\delta_2\rangle_{\vec{j}} \otimes \dots \otimes |\delta_n\rangle_{\vec{j}}) \end{aligned} \quad (331)$$

where $\left| \vec{j} \cdot a, \tau, \sigma \right\rangle$ denotes wave-function of the bare knot that corresponds to fermionic excitation of 2-level Dirac knot-crystal described by $\mathbf{Z}_{\text{Dirac}}(\vec{x}, t)$. \vec{j} labels the position of the composite knot, $\tau = A/B$ labels the chiral index (for the case of $\tau = A$ we have a knot-solution with $\eta_{A,\theta}(x'_1) > \eta_{B,\theta}(x'_1)$ and for the case of $\tau = B$, we have $\eta_{A,\theta}(x'_1) < \eta_{B,\theta}(x'_1)$), $\sigma = \uparrow / \downarrow$ labels spin-degrees of freedom. $|\delta_1\rangle_{\vec{j}} \otimes |\delta_2\rangle_{\vec{j}} \otimes \dots \otimes |\delta_n\rangle_{\vec{j}}$ describes the wave-function of n -internal-winding of the substrate knot-crystal described by $\mathbf{Z}_{\text{substrate}}(\vec{x}, t)$ as

$$\begin{aligned} & |\delta_1\rangle_{\vec{j}} \otimes |\delta_2\rangle_{\vec{j}} \otimes \dots \otimes |\delta_n\rangle_{\vec{j}} \\ &= |\delta_1\rangle_{\vec{j}} \otimes |\delta_2\rangle_{\vec{j}} \dots \otimes |\delta_i\rangle_{\vec{j}} \dots \otimes |\delta_n\rangle_{\vec{j}} \end{aligned} \quad (332)$$

where δ_i labels the existence of an internal-winding inside a given composite knot. For $\delta_i = 1$, we have the i -th internal-winding inside a given composite knot; for $\delta_i = 0$, we don't.

Due to spatial translation symmetry of composite knot-crystal, to get an arbitrary quantum state, we summarize quantum states on different "lattice" sites, i.e., $\left| \vec{j} \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle$, $\left| (\vec{j} + \vec{e}) \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle$, $\left| (\vec{j} + 2\vec{e}) \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle$, ... Therefore, we have the wave-function for an arbitrary quantum state to be

$$\prod_{\vec{j}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n} \left| \vec{j} \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle. \quad (333)$$

b. The ground state The ground state $|\text{vac}\rangle$ of a composite knot-crystal is denoted by

$$\begin{aligned} |\text{vac}\rangle &= \prod_{\vec{j}, \tau, \sigma} \left| \vec{j} \cdot a, \tau, \sigma, 1, 1, \dots, 1 \right\rangle \\ &= \prod_{\vec{j}, \tau, \sigma} \left| \vec{j} \cdot a, \tau, \sigma \right\rangle \\ &\quad \otimes (|\delta_1\rangle_{\vec{j}} \otimes |\delta_2\rangle_{\vec{j}} \otimes \dots \otimes |\delta_n\rangle_{\vec{j}}), \end{aligned} \quad (334)$$

where $\delta_{1\vec{j}} = \delta_{2\vec{j}} = \dots = \delta_{n\vec{j}} = 1$ denotes the filled internal-windings for the ground state at x , $x + \vec{j} \cdot a/n$, ..., $x + \vec{j} \cdot (n-1)a/n$, respectively. There exist global and internal spacial translation symmetries for the ground state, of which all internal-windings $|\delta_i\rangle_{\vec{j}}$ have uniform distribution. For the case of $n = 3$, the ground state is shown in Fig.14.

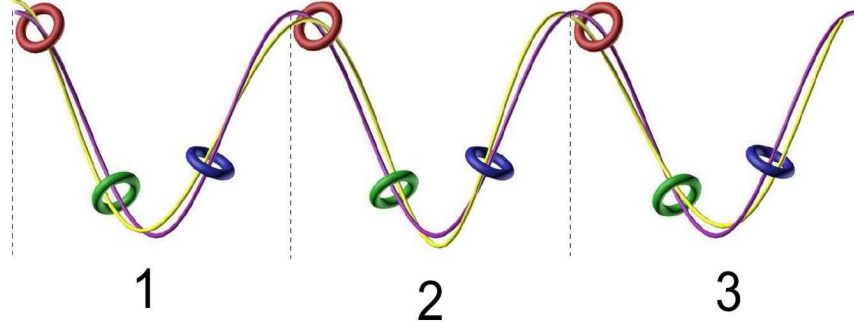


FIG. 14: An illustration of the ground state of a composite knot-crystal with 3 internal-windings, that is a 2-level double-helix knot-crystal with $(\Delta_{12})_{A/B} = (\Delta_{12})_B = 3 + \frac{1}{2}$. This composite knot-crystal can be regarded as a combination as a 2-level double-helix knot-crystal with $(\Delta_{12})_{A/B} = (\Delta_{12})_B = \frac{1}{2}$ and a substrate 2-level double-helix knot-crystal with $(\Delta_{12})_{A/B} = (\Delta_{12})_B = 3$. The two lines denote two 1-branes and each small ring denotes an internal-winding.

c. Quantum excited states for one composite knot In general, all quantum states beyond the ground state are quantum excited states for composite knots. For a localized quantum state of composite knot, the wave-function is just $|\vec{j} \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n\rangle$. We classify quantum excited states by counting the number of internal-windings of a composite knot. There are $n + 1$ -types of composite knots, of which the number of internal-windings is 0, 1, 2, ..., n . We define the number of internal-windings of a composite knot to be n_k . A bare knot without internal-winding corresponds to neutrinos, of which the number of internal-windings is always zero $n_k = n_{\nu_e} \equiv 0$. A composite knot with n -internal-winding corresponds to electrons, of which the number of internal-windings is always n ($n_k = n_e \equiv n$). Other types of composite knots correspond to quarks, of which the number of internal-windings is $n_k = n_{\text{quark}} = 1, 2, \dots, n - 1$.

For the case of $n = 1$, we have two types of knots, neutrino and electron. A neutrino is a bare knot without internal-winding ($n_k = n_{\nu_e} \equiv 0$)

$$|\vec{j} \cdot a, \tau, \sigma, \delta = 0\rangle = |\vec{j} \cdot a, \tau, \sigma\rangle_{\nu_e} \quad (335)$$

and an electron is a composite knot with 1-winding ($n_k = n_e \equiv 1$)

$$|\vec{j} \cdot a, \tau, \sigma, \delta = 1\rangle = |\vec{j} \cdot a, \tau, \sigma\rangle_e. \quad (336)$$

To describe the quantum excited states more clear, we introduce the operator representation,

$$\begin{aligned} |\vec{j} \cdot a, \tau, \sigma\rangle_{\nu_e} &= (\nu_e)_{\vec{j}, \tau, \sigma}^\dagger |\text{vac}\rangle \\ |\vec{j} \cdot a, \tau, \sigma\rangle_e &= e_{\vec{j}, \tau, \sigma}^\dagger |\text{vac}\rangle. \end{aligned} \quad (337)$$

For the case of $n = 2$, we have three types of knots. Neutrino is bare knot without internal-winding ($n_k = n_{\nu_e} \equiv 0$)

$$\begin{aligned} |\vec{j} \cdot a, \tau, \sigma, \delta_1 = 0, \delta_2 = 0\rangle &= |\vec{j} \cdot a, \tau, \sigma\rangle_{\nu_e} \\ &= (\nu_e)_{\vec{j}, \tau, \sigma}^\dagger |\text{vac}\rangle. \end{aligned} \quad (338)$$

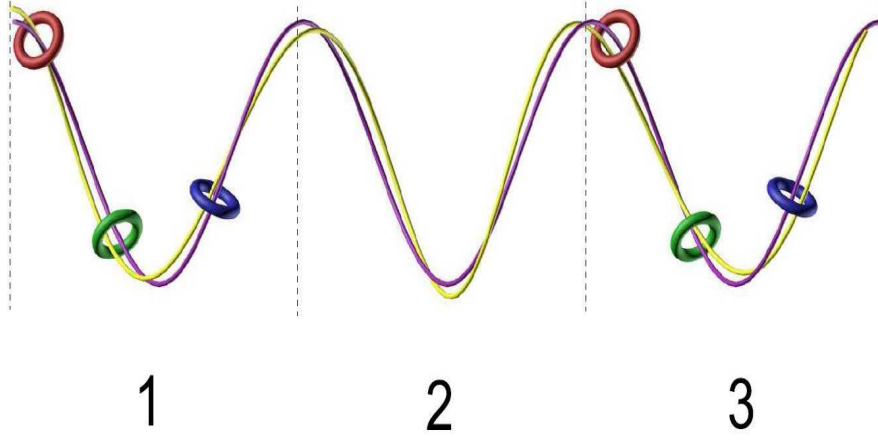


FIG. 15: An illustration of a neutrino at site 2 by removing 3 internal-windings ($n_{\nu_e} = 0$). The two lines denote two 1-branes and each small ring denotes an internal-winding.

Quarks are composite knots with 1 internal-winding ($n_k = n_{\text{quark}} = 1$), of which the two degenerate internal states are described by

$$\begin{aligned} \left(\begin{array}{c} \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 0, \delta_2 = 1 \right\rangle \\ \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 1, \delta_2 = 0 \right\rangle \end{array} \right) &= \left(\begin{array}{c} \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{1,\text{quark}} \\ \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{2,\text{quark}} \end{array} \right) \\ &= \left(\begin{array}{c} q_{1,\vec{j},\tau,\sigma} |\text{vac}\rangle \\ q_{2,\vec{j},\tau,\sigma} |\text{vac}\rangle \end{array} \right). \end{aligned} \quad (339)$$

Electrons are composite knots with 2 internal-windings ($n_k = n_{\nu_e} \equiv 0$) that is described by the state

$$\begin{aligned} \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 1, \delta_2 = 1 \right\rangle &= \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_e \\ &= e_{\vec{j},\tau,\sigma}^\dagger |\text{vac}\rangle. \end{aligned} \quad (340)$$

For the case of $n = 3$, we have four types of knots, of which the ground state is shown in Fig.14. Neutrino is bare knot without internal-winding ($n_k = n_{\nu_e} = 0$)

$$\begin{aligned} &\left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 0, \delta_2 = 0, \delta_3 = 0 \right\rangle \\ &= \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{\nu_e} = (\nu_e)_{\vec{j},\tau,\sigma}^\dagger |\text{vac}\rangle. \end{aligned} \quad (341)$$

See the illustration in Fig.15. In Fig.15, a neutrino is excited at site-2.

The d-quarks are composite knots with 1-internal-winding ($n_k = n_{\text{d-quark}} = 1$), of which there are three degenerate

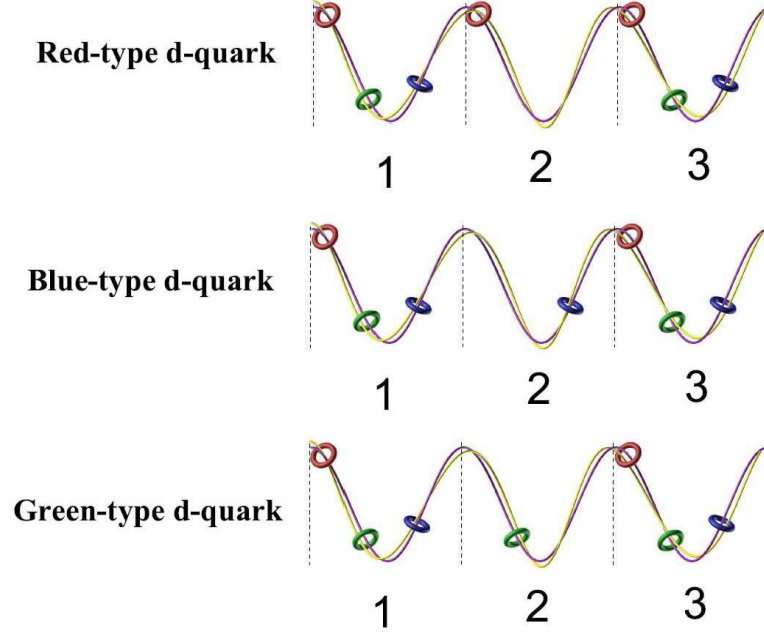


FIG. 16: An illustration of three degenerate states of a d-quark with 1 internal-winding ($n_{d\text{-quark}} = 1$) at site-2, that correspond to $|\vec{j} \cdot a, \tau, \sigma\rangle_{1,d\text{-quark}}$, $|\vec{j} \cdot a, \tau, \sigma\rangle_{2,d\text{-quark}}$, $|\vec{j} \cdot a, \tau, \sigma\rangle_{3,d\text{-quark}}$. The two lines denote two 1-branes and each small ring denotes an internal-winding.

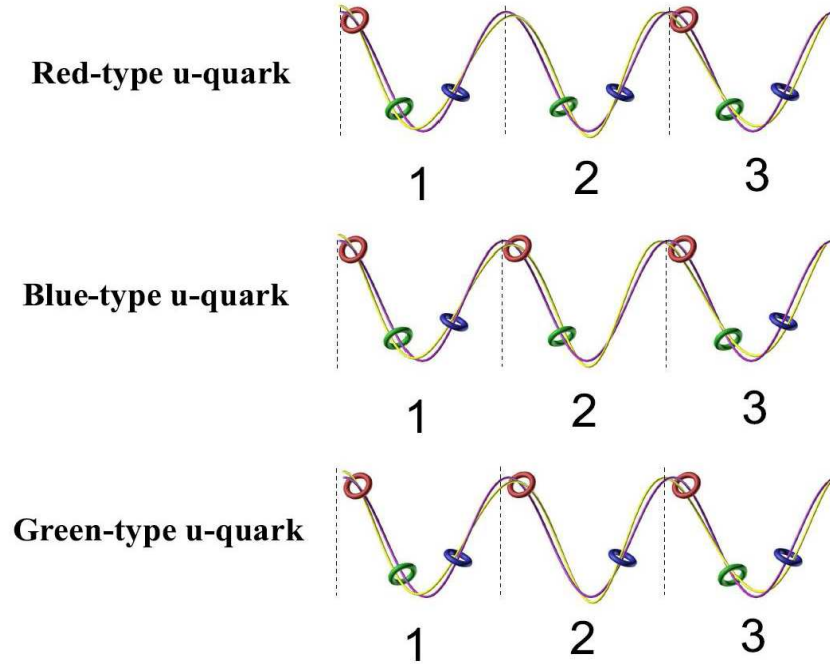


FIG. 17: An illustration of three degenerate states of a u-quark with 2 internal-windings ($n_{u\text{-quark}} = 2$) at site-2, that correspond to $|\vec{j} \cdot a, \tau, \sigma\rangle_{1,u\text{-quark}}$, $|\vec{j} \cdot a, \tau, \sigma\rangle_{2,u\text{-quark}}$, $|\vec{j} \cdot a, \tau, \sigma\rangle_{3,u\text{-quark}}$. The two lines denote two 1-branes and each small ring denotes an internal-winding.

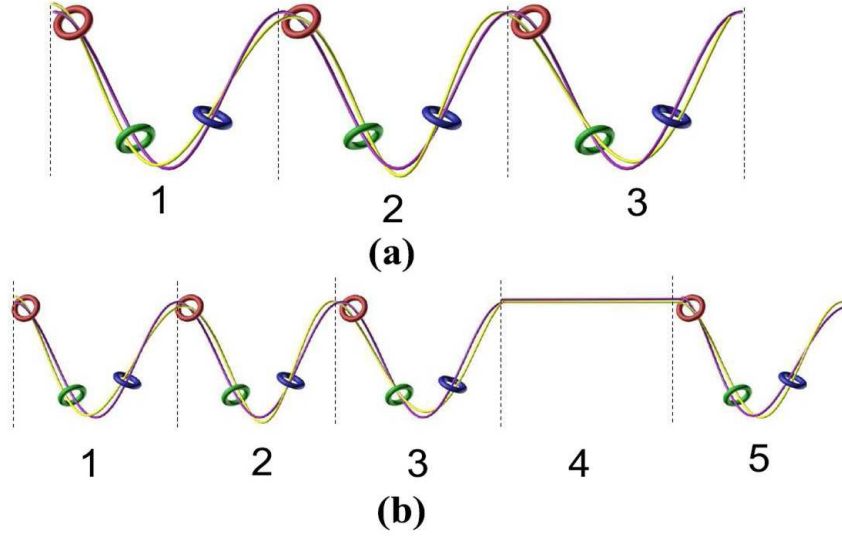


FIG. 18: An illustration of an electron with $n_e = 3$ by removing a ring at site-4. The two lines denote two 1-branes and each small ring denotes an internal-winding.

states described by

$$\begin{aligned}
 & \begin{pmatrix} \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 1, \delta_2 = 0, \delta_3 = 0 \right\rangle \\ \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 0, \delta_2 = 1, \delta_3 = 0 \right\rangle \\ \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 0, \delta_2 = 0, \delta_3 = 1 \right\rangle \end{pmatrix} \\
 &= \begin{pmatrix} \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{1,d\text{-quark}} \\ \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{2,d\text{-quark}} \\ \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{3,d\text{-quark}} \end{pmatrix} = \begin{pmatrix} d_{1,\vec{j},\tau,\sigma} |\text{vac}\rangle \\ d_{2,\vec{j},\tau,\sigma} |\text{vac}\rangle \\ d_{3,\vec{j},\tau,\sigma} |\text{vac}\rangle \end{pmatrix}. \tag{342}
 \end{aligned}$$

See the illustration in Fig.16. In Fig.16, a d-quark with three internal degenerate states is excited at site-2. The three degenerate states of d-quarks are called red d-quark, blue d-quark and green d-quark, respectively.

The u-quarks are composite knots with 2 internal-windings ($n_k = n_{u\text{-quark}} = 2$), of which the three degenerate states are described by

$$\begin{aligned}
 & \begin{pmatrix} \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 0, \delta_2 = 1, \delta_3 = 1 \right\rangle \\ \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 1, \delta_2 = 0, \delta_3 = 1 \right\rangle \\ \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 1, \delta_2 = 1, \delta_3 = 0 \right\rangle \end{pmatrix} \\
 &= \begin{pmatrix} \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{1,u\text{-quark}} \\ \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{2,u\text{-quark}} \\ \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{3,u\text{-quark}} \end{pmatrix} = \begin{pmatrix} u_{1,\vec{j},\tau,\sigma} |\text{vac}\rangle \\ u_{2,\vec{j},\tau,\sigma} |\text{vac}\rangle \\ u_{3,\vec{j},\tau,\sigma} |\text{vac}\rangle \end{pmatrix}. \tag{343}
 \end{aligned}$$

See the illustration in Fig.17. In Fig.17, a u-quark with three internal degenerate states is excited at site-2. The three degenerate states of u-quarks are called red u-quark, blue u-quark and green u-quark, respectively.

The electrons are composite knots with 3 internal-windings ($n_k = n_e = 3$), that is described by the state

$$\begin{aligned} & \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 1, \delta_2 = 1, \delta_3 = 1 \right\rangle \\ &= \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_e = e_{\vec{j}, \tau, \sigma}^\dagger |\text{vac}\rangle. \end{aligned} \quad (344)$$

See the illustration in Fig.18. In Fig.18, an anti-electron is excited at site-4.

d. Generalized translation symmetry for quantum states of composite knots We then consider a quantum excited state $\left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle$ where \vec{k} is the wave-vector of the waving brane-twisting. If we identify composite knots as elementary particles, there are two types of generalized translation symmetry for quantum states of composite knots, of which the translation operations are $\hat{T}_{\text{global}}(\Delta\vec{x})$ and $\hat{T}_{\text{internal}}(\Delta\vec{x})$, respectively.

Global spatial translation symmetry On the one hand, the global translation operation $\hat{\mathcal{T}}_{\text{internal}}(\Delta\vec{x})$ for composite knot-crystal turns into global spatial translation operation $\hat{T}_{\text{global}}(\Delta\vec{x})$ for internal quantum states of composite knots, i.e.,

$$\hat{\mathcal{T}}(\Delta\vec{x}) \implies \hat{T}_{\text{global}}(\Delta\vec{x}). \quad (345)$$

Under $\hat{T}_{\text{global}}(\Delta\vec{x})$, we have

$$\begin{aligned} & \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle \\ & \rightarrow \hat{T}_{\text{global}}(\Delta\vec{x}) \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle \\ &= e^{\pm i\vec{k}_0 \cdot [2a[(\Delta\vec{x}) \bmod 2a]]} e^{i\vec{k} \cdot [(\Delta\vec{x}) - 2a[(\Delta\vec{x}) \bmod 2a]]} \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle. \end{aligned} \quad (346)$$

For the case of $(\Delta\vec{x}) \bmod 2a = 0$, we have

$$\begin{aligned} & \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle \rightarrow \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle' \\ &= \hat{T}_{\text{global}}(\Delta\vec{x}) \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle \\ &= e^{i\vec{k} \cdot (\Delta\vec{x})} \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle. \end{aligned} \quad (347)$$

For the case of $(\Delta\vec{x}) \bmod 2a \neq 0$, we can recover the continuous spatial translation symmetry by changing projection angle $\theta \rightarrow \theta' = \theta \mp 2\pi[(\Delta\vec{x}) \bmod 2a]$.

Internal spatial translation symmetry On the other hand, the internal translation operation $\hat{\mathcal{T}}_{\text{internal}}(\Delta\vec{x})$ for composite knot-crystal turns into internal spatial translation operation $\hat{T}_{\text{internal}}(\Delta\vec{x})$ for internal quantum states of composite knots, i.e.,

$$\hat{\mathcal{T}}_{\text{internal}}(\Delta\vec{x}) \implies \hat{T}_{\text{internal}}(\Delta\vec{x}). \quad (348)$$

Under $\hat{T}_{\text{internal}}(\Delta\vec{x})$, we have

$$\hat{T}_{\text{internal}}(\Delta\vec{x}) \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle = \left| \vec{k}, \tau', \sigma, \delta'_1, \delta'_2, \dots, \delta'_n \right\rangle. \quad (349)$$

From $\hat{\mathcal{T}}_{\text{internal}}(\Delta\vec{x}) \implies \hat{T}_{\text{internal}}(\Delta\vec{x})$, there are three types of internal spatial translation symmetries:

1. For the case of $(\Delta\vec{x}) \bmod a = 0$, the internal translation symmetry for composite knots becomes trivial

$$\hat{T}_{\text{internal}}(\Delta\vec{x}) |\delta_i\rangle = |\delta_i\rangle \quad (350)$$

and

$$\begin{aligned} & \hat{T}_{\text{internal}}(\Delta\vec{x}) \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle \\ &= \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle; \end{aligned} \quad (351)$$

2. For the case of $(\Delta\vec{x}) \bmod a \neq 0$ and $(\Delta\vec{x}) \bmod (\frac{a}{n}) = 0$, there exists *discrete internal (D-internal) translation symmetry for internal-windings* and we have

$$|\delta_i\rangle \rightarrow \hat{T}_{D\text{-internal}}(\Delta\vec{x}) |\delta_i\rangle = |\delta_{i+1}\rangle, \quad (352)$$

$$\begin{aligned} & \hat{T}_{D\text{-internal}}(\Delta\vec{x}) \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_{n-1}, \delta_n \right\rangle \\ &= \left| \vec{k}, \tau, \sigma, \delta_2, \delta_3, \dots, \delta_n, \delta_1 \right\rangle \end{aligned} \quad (353)$$

and

$$\begin{aligned} & [\hat{T}_{D\text{-internal}}(\Delta\vec{x})]^n \left| x + \vec{j} \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle \\ & \rightarrow \left| x + \vec{j} \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle; \end{aligned} \quad (354)$$

3. For the case of $(\Delta\vec{x}) \bmod a \neq 0$ and $(\Delta\vec{x}) \bmod (\frac{a}{n}) \neq 0$, there exists *continuous internal (C-internal) translation symmetry for internal-windings* and we have

$$|\delta_i\rangle \rightarrow \hat{T}_{C\text{-internal}}(\Delta\vec{x}) |\delta_i\rangle = e^{i2\pi \cdot [(\Delta\vec{x}) \bmod (\frac{a}{n})]} |\delta_i\rangle \quad (355)$$

and

$$\begin{aligned} & \hat{T}_{C\text{-internal}}(\Delta\vec{x}) \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_{n-1}, \delta_n \right\rangle \\ &= e^{i2\pi \cdot [(\Delta\vec{x}) \bmod (\frac{a}{n})]} \left| \vec{k}, \tau, \sigma, \delta_2, \delta_3, \dots, \delta_n, \delta_1 \right\rangle. \end{aligned} \quad (356)$$

4. Emergent quantum field theory for 1-level composite knot-crystal

In above part, we have shown two types of spatial translation symmetries of the composite knot-crystal. In this part, we will show that the two types of spatial translation symmetries lead to different quantum structures for composite knot-crystal: for the case of $(\Delta\vec{x}) \bmod a = 0$, there exists generalized Bloch theorem for composite knots and the discrete global spatial translation symmetry ($\hat{T}_{\text{global}} |\psi\rangle = |\psi\rangle'$) leads to a Dirac model for composite knots (elementary fermions); for the case of $(\Delta\vec{x}) \bmod a \neq 0$ and $(\Delta\vec{x}) \bmod (\frac{a}{n}) \neq 0$, the discrete internal spatial translation symmetry ($\hat{T}_{D\text{-internal}} |\psi\rangle = |\psi\rangle'$) for internal-windings leads to an $SU(n)$ gauge symmetry for composite knots (elementary fermions) and for the case of $(\Delta\vec{x}) \bmod a \neq 0$ and $(\Delta\vec{x}) \bmod (\frac{a}{n}) = 0$, the continuous internal spatial translation symmetry ($\hat{T}_{C\text{-internal}} |\psi\rangle = |\psi\rangle'$) for internal-windings leads to a $U(1)$ gauge symmetry for composite knots (elementary fermions).

Based on the internal translation symmetry of composite knot-crystal, we show the physics picture of gauge symmetry. To well define a composite knot with n -internal-winding, we must choose arbitrary n nearest-neighbor internal-windings. Gauge symmetries appear as the redundancies to define knots. There are two types of redundancies to define composite knots: one corresponds to the exact position of an internal-winding inside a composite knot, the other is to choose which n -internal-winding to make up this composite knot. The first redundancy protected by continuous internal spatial translation symmetry for internal-windings leads to an Abelian gauge symmetry that is really the $U(1)$ gauge symmetry for electromagnetic interaction. The redundancy protected by discrete internal internal spatial translation symmetry for internal-windings leads to a non-Abelian gauge symmetry that is really the $SU(n)$ gauge symmetry for strong interaction. As a result, we have an $SU(n) \times U(1)$ gauge symmetry that will never be broken. The rule to settle down the redundancy is the rule to fix the gauge for gauge field.

The fluctuations that break down spatial translation symmetries become quasi-particles in quantum field theory. There are $n + 1$ types of composite knots that are topological excitations in a composite knot-crystal. Except for the topological excitations (the knots), there exist collective excitations – gauge fields. Position-fluctuations of internal-windings are Abelian gauge field and order-fluctuations of internal-windings are non-Abelian gauge field.

a. Emergent quantum field theory for composite knots In this part we discuss the emergent quantum field theory for composite knots without taking into account the detailed structures of internal-windings.

In mathematic, we consider the discrete global translation symmetry for composite knots. Thus we have

$$|\delta_i\rangle \rightarrow |\delta_i\rangle' = \hat{T}_{\text{global}}(\Delta\vec{x}) |\delta_i\rangle = |\delta_i\rangle \quad (357)$$

and

$$\begin{aligned}
& \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle \rightarrow \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle' \\
& = \hat{T}_{\text{global}}(\Delta \vec{x}) \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle \\
& = e^{i\vec{k} \cdot (\Delta \vec{x})} \left| \vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle
\end{aligned} \tag{358}$$

where $(\Delta \vec{x}) \bmod a = 0$. There exists the generalized Bloch theorem for all types of composite knots. The low energy quantum field theory for composite knots also becomes Dirac model.

For a composite knot-crystal, there also exist two types of energy costs – energy cost from curving a brane and that from rotating the knot-crystal. To characterize the energy cost from brane-curving, we define the coupling constant J between two nearest-neighbor composite knots. To characterize the energy cost from rotating of knot-crystal, we assume the rotating velocity to be $\omega = |\omega_A| = |\omega_B|$.

Firstly, we define the energy cost from brane-curving between two nearest-neighbor composite knots to be J and use the following formulation to characterize this energy cost,

$$\begin{aligned}
& \frac{J}{2} \left| x + \vec{j} \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle \\
& \cdot \left\langle x + (\vec{j} + \vec{e}) \cdot a, -\tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right| + h.c.,
\end{aligned} \tag{359}$$

where $\left| x + \vec{j} \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle$ and $\left| x + (\vec{j} + \vec{e}) \cdot a, -\tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle$ denote the quantum states of two nearest-neighbor composite knots. Because we assume that J mainly comes from the brane-curving for 2-th level windings, in the limit of $r_{A,1} = r_{B,1} = r_1 \ll r_{A,2} = r_{B,2} = r_2$, different types of composite knots have the same J . This fact guarantees a universal light speed

$$c \equiv aJ. \tag{360}$$

By using the concept of "fragmentized knot" and considering the generalized translation symmetry of the composite knot-crystal, the Planck constant for different types of composite knots is same as

$$\hbar = \frac{a \cdot k_B T_{\text{brane}}}{((\Delta_{12})_{A/B}) 3\pi \cdot c_{5D}} = \frac{a_1 \cdot k_B T_{\text{brane}}}{3\pi \cdot c_{5D}} \tag{361}$$

where $a_1 = \frac{a}{(\Delta_{12})_{A/B}}$ is the half pitch of 1-th winding of the 2-level knot-crystal.

For simplicity, we take the case of $n = 3$ as an example. By introducing operator representation, we use the traditional effective Hamiltonian to describe the dynamics of composite knot-crystal as

$$\hat{\mathcal{H}}_{\text{coupling}} = \frac{J}{2} \sum_{\langle ij \rangle} \psi_i^\dagger \psi_j + h.c. \tag{362}$$

where

$$\psi_i^\dagger = \begin{pmatrix} (\nu_e)_i^\dagger \\ e_i^\dagger \\ u_i^\dagger \\ d_i^\dagger \end{pmatrix} \tag{363}$$

and

$$u_i^\dagger = \begin{pmatrix} u_{1,i}^\dagger \\ u_{2,i}^\dagger \\ u_{3,i}^\dagger \end{pmatrix}, \tag{364}$$

$$d_i^\dagger = \begin{pmatrix} d_{1,i}^\dagger \\ d_{2,i}^\dagger \\ d_{3,i}^\dagger \end{pmatrix}. \tag{365}$$

After considering the rotating effect, we write down the low energy effective Lagrangian as

$$\begin{aligned}\mathcal{L} &= i \sum_i \psi_i^\dagger \partial_t \psi_i' - \hat{\mathcal{H}}_{\text{coupling}} \\ &= i \int \psi^\dagger \partial_t \psi d^3x - \hat{\mathcal{H}}_{3\text{D}}\end{aligned}\quad (366)$$

where

$$\begin{aligned}\hat{\mathcal{H}}_{3\text{D}} &= \hat{\mathcal{H}}_{\text{coupling}} + m \int \psi^\dagger (\tau_x \otimes 1) \psi d^3x \\ &= c \int \psi^\dagger \{[(\tau_y \otimes \vec{\sigma}) \cdot \hat{k}] \otimes 1\} \psi d^3x \\ &\quad + \int \psi^\dagger [\tau_x \otimes \mathbf{1} \otimes \mathbf{m}] \psi d^3x\end{aligned}\quad (367)$$

where the term $\psi^\dagger \{[(\tau_y \otimes \vec{\sigma}) \cdot \hat{k}] \otimes 1\} \psi$ denotes the process of brane-curving; the term $\psi^\dagger [\tau_x \otimes \mathbf{1} \otimes \mathbf{m}] \psi$ denotes the

process of brane-rotating. $\mathbf{1}$ is a two-by-two matrix, 1 is four-by-four unit matrix and $\mathbf{m} = \begin{pmatrix} m_{\nu_e} & 0 & 0 & 0 \\ 0 & m_e & 0 & 0 \\ 0 & 0 & m_u & 0 \\ 0 & 0 & 0 & m_d \end{pmatrix}$ is

four-by-four mass matrix. Without considering confinement effect, due to the same rotating velocity, all fermionic elementary particles have the same mass

$$m_e c^2 = m_{\nu_e} c^2 = m_d c^2 = m_u c^2 = \hbar \omega. \quad (368)$$

Finally, the Lagrangian of Dirac fermion of composite knot-crystal is

$$\begin{aligned}\mathcal{L}_{\text{fermion}}(x) &= \bar{\nu}_e(x) i \gamma^\mu \partial_\mu \nu_e(x) + \bar{e}(x) i \gamma^\mu \partial_\mu e(x) \\ &\quad + \bar{d}(x) i \gamma^\mu \partial_\mu d(x) + \bar{u}(x) i \gamma^\mu \partial_\mu u(x) \\ &\quad + m_e \bar{e}(x) e(x) + m_{\nu_e} \bar{\nu}_e(x) \nu_e(x) \\ &\quad + m_d \bar{d}(x) d(x) + m_u \bar{u}(x) u(x).\end{aligned}\quad (369)$$

In particular, the neutrino has both left and right hands and all fermionic elementary particles have the same mass.

b. Emergent quantum field theory for U(1) gauge field Due to the redundancy protected by continuous internal spatial translation symmetry for internal-windings, there exists U(1) gauge symmetry to define a given internal-winding $|\delta_i\rangle_{\vec{j}}$. The U(1) gauge symmetry indicates that we may locally reorganize the knots by different ways and get the same result. Fig.19 is an illustration of two different gauges for U(1) gauge symmetry, of which the positions of internal-windings are different.

In mathematic, the redundancy of U(1) gauge symmetry comes from the continuous internal spatial translation symmetry. For the case of $(\Delta \vec{x}) \bmod a \neq 0$ and $(\Delta \vec{x}) \bmod \left(\frac{a}{n}\right) \neq 0$, we have

$$\hat{T}_{\text{C-intenal}}(\Delta \vec{x}) |\delta_i\rangle = e^{i\Delta\phi} |\delta_i\rangle \quad (370)$$

where

$$\Delta\phi = 2\pi \cdot [(\Delta \vec{x}) \bmod \left(\frac{a}{n}\right)]. \quad (371)$$

As a result, under a local shifting, the quantum phase from an internal-winding $|\delta_i\rangle$ changes. The definition of an internal-winding $|\delta_i\rangle$ locally depends on the phase rule $\Delta\phi_{\vec{j}}$ that indicates a local U(1) gauge transformation,

$$|\delta_i\rangle_{\vec{j}} \rightarrow \hat{T}_{\text{C-intenal}}(\Delta \vec{x}) |\delta_i\rangle_{\vec{j}} = |\delta_i\rangle_{\vec{j}+\Delta \vec{x}} e^{i\Delta\phi_{\vec{j}}}. \quad (372)$$

We may summarize the emergence of U(1) gauge symmetry as

$$\begin{aligned}\text{Continuous internal translation symmetry} \\ \implies \text{U(1) gauge symmetry.}\end{aligned}\quad (373)$$

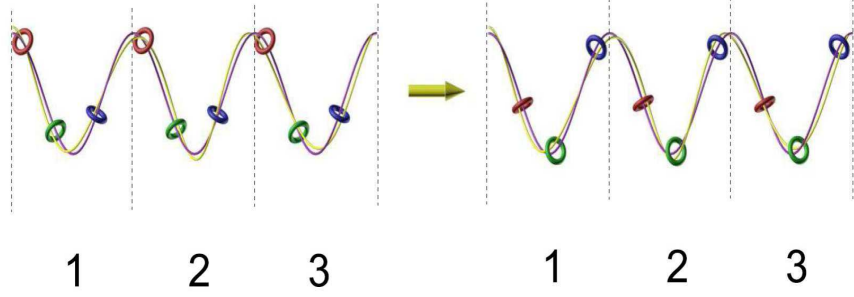


FIG. 19: An illustration of two different gauges of U(1) gauge symmetry, of which the positions of internal-windings are different. 1, 2, 3 label three composite knots and the small (red, green, blue) rings denote the three types of internal-windings.

We then show the physics picture of U(1) gauge symmetry in composite knot-crystal by comparing it to Kaluza–Klein theory[24]. Each internal-winding is really a 2π -flux in extra space (x_{d+1} - x_{d+2} space) and the position-fluctuations of an internal-winding cause quantum phase-fluctuations of composite knots. When the internal-windings shift $\vec{j} \rightarrow \vec{j} + \Delta\vec{x}$, the phase of the corresponding composite knot changes, $\psi_{\vec{j}} \rightarrow \psi_{\vec{j}} e^{i\Delta\phi_{\vec{j}}}$ where $\Delta\phi_{\vec{j}} = 2\pi \cdot [(\Delta\vec{x}) \bmod (\frac{a}{n})]$. Now the electric charge e_0 is $\frac{a}{n}$. For composite knot with n -internal-winding, the total flux in extra space is $2\pi n$, of which the total electric charge is $n\frac{a}{n} = ne_0$. The situation is very similar to the emergent gauge symmetry in Kaluza–Klein theory, in which the particles have finite angular momentum m in extra space that corresponds to n -internal-winding in knot theory.

After considering the local changing of phase angle, we have the effective Hamiltonian from brane-curved,

$$\begin{aligned}
 & J \sum_{\langle \vec{j}, \vec{j}' \rangle} \left| \vec{j} \cdot a + \phi_{\vec{j}} \cdot a/\pi, A, \uparrow, \delta_1, \delta_2, \dots, \delta_n \right\rangle e^{i\Delta\phi_{\vec{j}} (\sum_{i=1}^n \delta_i)} \\
 & \cdot e^{i\Delta\phi_{\vec{j}'} (\sum_{i=1}^n \delta_i)} \left\langle \vec{j}' \cdot a + \phi_{\vec{j}'} \cdot a/\pi, B, \downarrow, \delta_1, \delta_2, \dots, \delta_n \right| + h.c. \\
 & = J \sum_{\langle \vec{j}, \vec{j}' \rangle} \left| \vec{j} \cdot a + \phi_{\vec{j}} \cdot a/\pi, A, \uparrow, \delta_1, \delta_2, \dots, \delta_n \right\rangle \\
 & \cdot e^{iA_{\vec{j}, \vec{j}'}} \left\langle \vec{j}' \cdot a + \phi_{\vec{j}'} \cdot a/\pi, B, \downarrow, \delta_1, \delta_2, \dots, \delta_n \right| + h.c.
 \end{aligned} \tag{374}$$

where

$$\begin{aligned}
 A_{\vec{j}, \vec{j}'} &= (\sum_{i=1}^n \delta_i) \times (\Delta\phi_{\vec{j}} - \Delta\phi_{\vec{j}'}) \\
 &= \vec{A}_{\vec{j}, \vec{j}+\vec{e}} \cdot \vec{e}
 \end{aligned} \tag{375}$$

and

$$\vec{A}_{\vec{j}, \vec{j}+\vec{e}} = (\sum_{i=1}^n \delta_i) \times (\phi_{\vec{j}} - \phi_{\vec{j}+\vec{e}}) \vec{e}. \tag{376}$$

The gauge field $\vec{A}_{\vec{j}, \vec{j}+\vec{e}}$ characterizes the local position perturbation of internal-windings in composite knot-crystal. In continuum limit, we have $\vec{A}_{\vec{j}, \vec{j}+\vec{e}} \rightarrow \vec{A}(x)$.

On the other hand, there exists local single occupation condition

$$\psi_j^\dagger \psi_j = 1. \quad (377)$$

To characterize the constraint, we introduce a Lagrangian variable $A_{0,\vec{j}}$ to path-integral formulation as

$$A_{0,\vec{j}} \psi_j^\dagger \psi_j. \quad (378)$$

In continuum limit, we have $A_{0,\vec{j}} \rightarrow A_0(x)$.

Finally, we derive the path-integral formulation to characterize the effective Hamiltonian in continuum limit for gauge field

$$\int \mathcal{D}A_0 \mathcal{D}\vec{A} e^{i\mathcal{S}_{\text{EM}}/\hbar} \quad (379)$$

where $\mathcal{S}_{\text{EM}} = \int \mathcal{L}_{\text{EM}} dt d^3x$ and

$$\mathcal{L}_{\text{EM}} = e A_\mu j_{(em)}^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (380)$$

The gauge field strength $F_{\mu\nu}$ is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The electric charge for an internal-winding is $e_0 = \frac{n}{a}$. As a result, the total electric charge of a composite knot with n -internal-winding is

$$e = (\sum_{i=1}^n \delta_i) e_0. \quad (381)$$

From Eq.380, we can derive the Maxwell equations $\partial_\mu F^{\mu\nu} = j_{(em)}^\nu$ where $j_{(em)}^\mu$ is the electric current. In addition, to give a correct definition of knots in a composite knot-crystal, we must set down the "gauge" $f(A_\mu) = 0$ that leads to a fixed $\phi(x)$ at a give position x .

For the case of $n = 3$, we derive the U(1) gauge theory for composite knot-crystal. Now, the charge of an electron is $n_e \cdot e_0 = 3e_0$; the charge of a u-quark is $n_{\text{u-quark}} \cdot e_0 = 2e_0$, the charge of a d-quark is $n_{\text{d-quark}} \cdot e_0 = e_0$, the charge of a neutrino is zero, $n_{\nu_e} \cdot e_0 = 0$. We define the electric charge for electrons to be unit as $3e_0 = e$. The electric charges for u-quark and d-quark are $e/3$ and $2e/3$, respectively. As a result, the effective Lagrangian for knots turns into

$$\begin{aligned} \mathcal{L}_{\text{fermion}}(x) = & \bar{\nu}_e(x) i\gamma^\mu \partial_\mu \nu_e(x) + \bar{e}(x) i\gamma^\mu \partial_\mu e(x) \\ & + \bar{d}(x) i\gamma^\mu \partial_\mu d(x) + \bar{u}(x) i\gamma^\mu \partial_\mu u(x) \\ & + m_e \bar{e}(x) e(x) + m_{\nu_e} \bar{\nu}_e(x) \nu_e(x) \\ & + m_d \bar{d}(x) d(x) + m_u \bar{u}(x) u(x) + e A_\mu(x) j_{(em)}^\mu(x) \end{aligned} \quad (382)$$

where

$$\begin{aligned} j_{(em)}^\mu(x) = & i\bar{e}(x) \gamma^\mu e(x) + i\frac{2}{3} \bar{u}(x) \gamma^\mu u(x) \\ & + i\frac{1}{3} \bar{d}(x) \gamma^\mu d(x). \end{aligned} \quad (383)$$

Finally, we show the physical picture of the U(1) gauge field. U(1) gauge field comes from phase fluctuations of internal-windings. Two composite knots interact by exchanging quantum fluctuations of n -internal-winding. Finite electronic field comes from density changing of internal-windings that corresponds to longitudinal tension of the composite knot-crystal. An extra internal-winding plays the role of source of electric field. When there exists an extra internal-winding, the composite knots will be attractive or repulsive to the source. On the other hand, the transverse tension of the composite knot-crystal corresponds to magnetic field. When there exists finite magnetic field, the path of moving composite knots will be curved.

We may summarize the emergence of U(1) gauge field as

$$\begin{aligned} & \text{Position-fluctuations of } n\text{-internal-winding} \\ \implies & \text{U(1) gauge field.} \end{aligned} \quad (384)$$

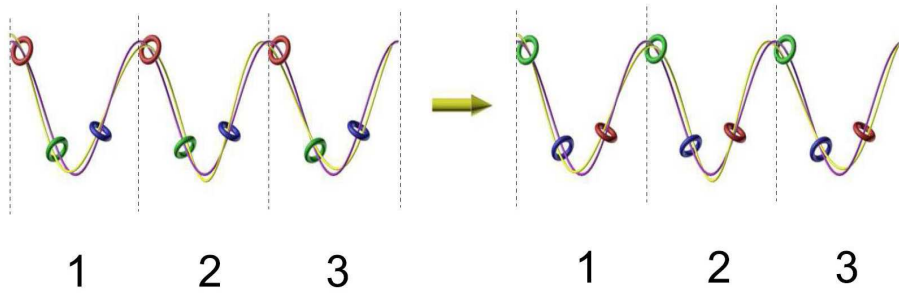


FIG. 20: An illustration of two different gauges of SU(3) gauge symmetry, of which the orders of three internal-windings are different. 1, 2, 3 label the composite knots and the small (red, green, blue) rings denote the three types of internal-windings.

c. Emergent quantum field theory for SU(n) gauge field In this part we discuss SU(n) non-Abelian gauge field of a composite knot-crystal.

For a quantum excited state of a composite knot-crystal described by $|\vec{k}, \tau, \sigma\rangle$, there exists discrete internal translation symmetry for internal-windings. For the case of $(\Delta\vec{x}) \bmod a \neq 0$ and $(\Delta\vec{x}) \bmod (\frac{a}{n}) = 0$, we have

$$|\delta_i\rangle \rightarrow \hat{T}_{D\text{-internal}}(\Delta\vec{x}) |\delta_i\rangle = |\delta_{i+1}\rangle \quad (385)$$

and

$$\begin{aligned} & \hat{T}_{D\text{-internal}}(\Delta\vec{x}) |\vec{k}, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_{n-1}, \delta_n\rangle \\ &= |\vec{k}, \tau, \sigma, \delta_2, \delta_3, \dots, \delta_n, \delta_1\rangle. \end{aligned} \quad (386)$$

According to discrete internal translation symmetry for internal-windings, all $|\delta_i\rangle_{\vec{j}}$ are same and the definition of the index i of an internal-winding $|\delta_i\rangle_{\vec{j}}$ is not unique. We can re-order the index of internal-windings by

$$\begin{aligned} & \hat{T}_{D\text{-internal}}(\Delta\vec{x}) |x + \vec{j} \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n\rangle \\ & \rightarrow |x + \vec{j} \cdot a, \tau, \sigma, \delta_2, \delta_3, \dots, \delta_n, \delta_1\rangle \end{aligned} \quad (387)$$

or

$$\begin{aligned} & [\hat{T}_{D\text{-internal}}(\Delta\vec{x})]^2 |x + \vec{j} \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n\rangle \\ & \rightarrow |x + \vec{j} \cdot a, \tau, \sigma, \delta_3, \delta_4, \dots, \delta_1, \delta_2\rangle, \end{aligned} \quad (388)$$

or

$$\begin{aligned} & [\hat{T}_{D\text{-internal}}(\Delta\vec{x})]^3 |x + \vec{j} \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n\rangle \\ & \rightarrow |x + \vec{j} \cdot a, \tau, \sigma, \delta_4, \delta_5, \dots, \delta_2, \delta_3\rangle \end{aligned} \quad (389)$$

or

...

In particular, we have

$$\begin{aligned} & [\hat{T}_{D-\text{internal}}(\Delta\vec{x})]^n \left| x + \vec{j} \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle \\ & \rightarrow \left| x + \vec{j} \cdot a, \tau, \sigma, \delta_1, \delta_2, \dots, \delta_n \right\rangle. \end{aligned} \quad (390)$$

This feature leads to an $SU(n)$ gauge symmetry for quarks.

We take the case of $n = 3$ as an example to show the $SU(3)$ gauge symmetry. After re-ordering the internal-winding degrees of freedom, the three degenerate states of d-quarks

$$\begin{aligned} & \begin{pmatrix} \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 1, \delta_2 = 0, \delta_3 = 0 \right\rangle \\ \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 0, \delta_2 = 1, \delta_3 = 0 \right\rangle \\ \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 0, \delta_2 = 0, \delta_3 = 1 \right\rangle \end{pmatrix} \\ & = \begin{pmatrix} \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{1,d\text{-quark}} \\ \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{2,d\text{-quark}} \\ \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{3,d\text{-quark}} \end{pmatrix} = \begin{pmatrix} d_{1,\vec{j},\tau,\sigma} |\text{vac}\rangle \\ d_{2,\vec{j},\tau,\sigma} |\text{vac}\rangle \\ d_{3,\vec{j},\tau,\sigma} |\text{vac}\rangle \end{pmatrix} \end{aligned} \quad (391)$$

can be transformed each other. For u-quarks, we have the similar result

$$\begin{aligned} & \begin{pmatrix} \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 0, \delta_2 = 1, \delta_3 = 1 \right\rangle \\ \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 1, \delta_2 = 0, \delta_3 = 1 \right\rangle \\ \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 1, \delta_2 = 1, \delta_3 = 0 \right\rangle \end{pmatrix} \\ & = \begin{pmatrix} \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{1,u\text{-quark}} \\ \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{2,u\text{-quark}} \\ \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_{3,u\text{-quark}} \end{pmatrix} = \begin{pmatrix} u_{1,\vec{j},\tau,\sigma} |\text{vac}\rangle \\ u_{2,\vec{j},\tau,\sigma} |\text{vac}\rangle \\ u_{3,\vec{j},\tau,\sigma} |\text{vac}\rangle \end{pmatrix}. \end{aligned} \quad (392)$$

This result leads to the concept of non-Abelian gauge symmetry. Fig.20 is an illustration of two different gauges for $SU(3)$ gauge symmetry, of which the orders of 3-internal-winding can be reorganized. However, for electron, a composite knot with 3-internal-winding, that is described by the state

$$\begin{aligned} & \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 1, \delta_2 = 1, \delta_3 = 1 \right\rangle \\ & = \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_e = e_{\vec{j},\tau,\sigma}^\dagger |\text{vac}\rangle, \end{aligned} \quad (393)$$

there is no degenerate internal state and the orders of 3-internal-winding will never change the quantum states of an electron. For neutrino without internal-winding, that is described by the state

$$\begin{aligned} & \left| \vec{j} \cdot a, \tau, \sigma, \delta_1 = 0, \delta_2 = 0, \delta_3 = 0 \right\rangle \\ & = \left| \vec{j} \cdot a, \tau, \sigma \right\rangle_e = \nu_{e,\vec{j},\tau,\sigma}^\dagger |\text{vac}\rangle, \end{aligned} \quad (394)$$

there is also no degenerate internal state.

We then derive the formulation of the gauge theory for the quarks with local $SU(n)$ symmetry. For simplicity,

we denote the quarks with n internal degrees of freedom by $\Psi_{\text{quark}} = \begin{pmatrix} \psi_{1,\text{quark}} \\ \psi_{2,\text{quark}} \\ \dots \\ \psi_{n,\text{quark}} \end{pmatrix}$ where $\psi_{i,\text{quark}}$ describes the

quantum state of the composite knots. According to continuous internal translation symmetry ($\hat{T}_{\text{D-internal}}(\Delta\vec{x})|\delta_i\rangle = |\delta_{i+1}\rangle$), $\psi_{i,\text{quark}}$ and $\psi_{i',\text{quark}}$ can be changed into each other by choosing different local gauges. Thus, Ψ_{quark} can be transformed locally according to an n -dimensional representation,

$$\Psi_{\text{quark}} \rightarrow \Psi'_{\text{quark}} \rightarrow U(x)\Psi_{\text{quark}}, \quad (395)$$

where $U(x) = e^{i\Theta(x)}$ is the matrix of the representation of SU(n) group. $\Theta(x) = \sum_{a=1}^{n^2-1} \theta^a \tau^a$ and θ^a are a set of $n^2 - 1$ constant parameters, and τ^a are $n^2 - 1$ $n \times n$ matrices representing the $n^2 - 1$ generators of the Lie algebra of SU(n) [25]. We may summarize the emergence of SU(n) gauge symmetry as

$$\begin{aligned} &\text{Discrete internal translation symmetry} \\ &\rightarrow \text{SU}(n) \text{ gauge symmetry.} \end{aligned}$$

The Lagrangian density $\mathcal{L}_{\text{YM}}(\text{SU}(n))$ is invariant under the gauge transformations with an x -dependent Θ . To characterize the non-Abelian gauge symmetry, C. N. Yang and R. L. Mills in 1954 introduced “Yang-Mills field”, $\mathcal{A}_\mu(x) = \sum_{a=1}^{n^2-1} A_\mu^a(x) \tau^a \rightarrow A_\mu^a \tau^a$, $a = 1, \dots, n^2 - 1$ that belong to the adjoint representation of SU(n). The non-Abelian gauge symmetry is represented by

$$\Psi'_{\text{quark}} \rightarrow e^{i\Theta(x)} \Psi_{\text{quark}} \quad (396)$$

and

$$\mathcal{A}_\mu(x) \rightarrow e^{i\Theta(x)} \mathcal{A}_\mu(x) e^{-i\Theta(x)} + \frac{i}{g} \left(\partial_\mu e^{i\Theta(x)} \right) e^{-i\Theta(x)}. \quad (397)$$

The gauge strength is defined by $\mathcal{G}_{\mu\nu}$ as

$$\mathcal{G}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - ig [\mathcal{A}_\mu, \mathcal{A}_\nu] \quad (398)$$

or

$$\mathcal{G}_{\mu\nu} = G_{\mu\nu}^a t^a, \quad G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c. \quad (399)$$

The Lagrangian of Yang-Mills field can only be written as:

$$\mathcal{L}_{\text{YM}}(\text{SU}(n)) = -\frac{1}{2} \text{Tr} \mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu} + \text{Tr} J_{\text{YM}}^\mu \mathcal{A}_\mu \quad (400)$$

where $J_{\text{YM}}^\mu = i\bar{\Psi}_{\text{quark}} \gamma^\mu \Psi_{\text{quark}}$.

Finally, we show the physical picture of the SU(n) gauge field. An extra internal-winding also plays the role of source of SU(n) gauge field and carries color degree of freedom. There are n different colors. For example, for the case of SU(3), there are three colors called red, blue and green, respectively. Quantum fluctuations of SU(n) gauge field are order-fluctuations of internal-windings. The quantized modes of SU(n) gauge field are always called gluons that are the quantum fluctuations of internal-windings. Two colored composite knots interact by exchanging gluons. However, a composite object with $n_k = N \cdot n$ internal-windings (N is an integer number) are colorless. Because electron is a composite knot with n -internal-winding ($n_e \equiv n$) and neutrino is composite knot without internal-winding ($n_{\nu_e} \equiv 0$), the order-fluctuations reorganizing internal-windings will never affect electrons and neutrinos. So, electrons and neutrinos don't couple the gluons.

We may summarize the emergence of SU(n) gauge field as

$$\begin{aligned} &\text{Order-fluctuations of internal-windings} \\ &\rightarrow \text{SU}(n) \text{ gauge field.} \end{aligned} \quad (401)$$

d. Examples for emergent quantum field theories of composite knot-crystals A composite knot-crystal naturally gives rise to gauge bosons (such as photons) and fermions (such as electrons). Photons are vibrations of substrate knot-crystal with n -internal-winding, while fermions are composite knots. We get the effective Lagrangian for a composite knot-crystal as

$$\begin{aligned} \mathcal{L}((\Delta_{12})_{A/B} = n + \frac{1}{2}) &= \mathcal{L}_{\text{fermion}} \\ &+ \mathcal{L}_{\text{EM}}(\text{U}(1)) + \mathcal{L}_{\text{YM}}(\text{SU}(n)). \end{aligned} \quad (402)$$

The U(1) gauge field characterizes the interaction from the position-fluctuations of internal-windings, $\phi_0 \rightarrow \phi_0 + \Delta\phi_j$. The SU(n) gauge field characterizes the interaction from the order-fluctuations of internal-windings.

Physics of composite knot-crystal with $\Delta_{12} = 0.5$ For the composite knot-crystal with $\Delta_{12} = 0.5$, the (composite) knot becomes neutrino with $n_e = 0$ and the effective model is a free Dirac model, of which the effective Lagrangian is

$$\mathcal{L}((\Delta_{12})_{A/B} = \frac{1}{2}) = \bar{\nu}_e(x) i \gamma^\mu \partial_\mu \nu_e(x) \quad (403)$$

$$+ m_{\nu_e} \bar{\nu}_e(x) \nu_e(x), \quad (404)$$

where $m_{\nu_e} = \hbar\omega/c^2$.

Physics of composite knot-crystal with $\Delta_{12} = 1.5$ For the composite knot-crystal with $\Delta_{12} = 1.5$, the (composite) knots are electron with $n_e = 1$ and neutrino with $n_{\nu_e} = 0$. The effective model is U(1) gauge theory, of which the effective Lagrangian is given by

$$\begin{aligned} \mathcal{L}((\Delta_{12})_{A/B} = 1.5) &= \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{EM}}(\text{U}(1)) \\ &= \bar{\nu}_e(x) i \gamma^\mu \partial_\mu \nu_e(x) + \bar{e}(x) i \gamma^\mu \partial_\mu e(x) \\ &\quad + m_e \bar{e}(x) e(x) + m_{\nu_e} \bar{\nu}_e(x) \nu_e(x) \\ &\quad - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e A_\mu(x) j_{(em)}^\mu(x) \end{aligned} \quad (405)$$

where $j_{(em)}^\mu(x) = i \bar{e}(x) \gamma^\mu e(x)$. $m_{\nu_e} = \hbar\omega/c^2$ and $m_e = \hbar\omega/c^2$ are masses for neutrino and electron, respectively. The phonons $A_\mu(x)$ characterize the position-fluctuations of internal-windings.

Physics of composite knot-crystal with $\Delta_{12} = 2.5$ For the composite knot-crystal with $\Delta_{12} = 2.5$, the (composite) knots are electron with $n_e = 2$, neutrino with $n_{\nu_e} = 0$, quarks with $n_{\text{quark}} = 1$. The effective model is SU(2)⊗U(1) gauge theory, of which the effective Lagrangian is given by

$$\begin{aligned} \mathcal{L}((\Delta_{12})_{A/B} = 2.5) &= \mathcal{L}_{\text{fermion}} \\ &\quad + \mathcal{L}_{\text{strong}}(\text{SU}(2)) + \mathcal{L}_{\text{EM}}(\text{U}(1)) \\ &= \bar{\nu}_e(x) i \gamma^\mu \partial_\mu \nu_e(x) + \bar{e}(x) i \gamma^\mu \partial_\mu e(x) \\ &\quad + \bar{\psi}_{\text{quark}}(x) i \gamma^\mu \partial_\mu \psi_{\text{quark}}(x) \\ &\quad + m_e \bar{e}(x) e(x) + m_{\nu_e} \bar{\nu}_e(x) \nu_e(x) + m_{\text{quark}} \bar{\psi}_{\text{quark}}(x) \psi_{\text{quark}}(x) \\ &\quad - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e A_\mu(x) j_{(em)}^\mu(x) \\ &\quad - \frac{1}{2} \text{Tr} \mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu} + \text{Tr} J_{\text{YM}}^\mu \mathcal{A}_\mu \end{aligned} \quad (406)$$

where $m_{\nu_e} = \hbar\omega/c^2$, $m_{\text{quark}} = \hbar\omega/c^2$ and $m_e = \hbar\omega/c^2$ are masses for neutrino, quark and electron, respectively. The electric current is

$$j_{(em)}^\mu(x) = i \bar{e}(x) \gamma^\mu e(x) + \frac{1}{2} i \bar{\psi}_{\text{quark}}(x) \gamma^\mu \psi_{\text{quark}}(x). \quad (407)$$

The SU(2) color current is $J_{\text{YM}}^{a,\mu} = i \bar{\Psi}_{\text{quark}} \gamma^\mu T^a \Psi_{\text{quark}}$.

The U(1) gauge field characterizes the interaction from the position-fluctuations of internal-windings. The SU(2) gauge field characterizes the interaction from the order-fluctuations of internal-winding.

Physics of composite knot-crystal with $\Delta_{12} = 3.5$ For the composite knot-crystal with $\Delta_{12} = 3.5$, we have four types of composite knots: electron with $n_e = 3$, d-quarks with $n_{\text{d-quark}} = 1$, u-quarks with $n_{\text{u-quark}} = 2$, neutrino

with $n_{\nu_e} = 0$. The effective $SU(3) \otimes U(1)$ gauge theory, of which the Lagrangian is

$$\begin{aligned}
\mathcal{L}((\Delta_{12})_{A/B} = 3.5) &= \mathcal{L}_{\text{fermion}} \\
&+ \mathcal{L}_{\text{strong}}(SU(3)) + \mathcal{L}_{\text{EM}}(U(1)) \\
&= \bar{\nu}_e(x) i\gamma^\mu \partial_\mu \nu_e(x) + \bar{e}(x) i\gamma^\mu \partial_\mu e(x) \\
&+ \bar{d}(x) i\gamma^\mu \partial_\mu d(x) + \bar{u}(x) i\gamma^\mu \partial_\mu u(x) \\
&+ m_e \bar{e}(x) e(x) + m_{\nu_e} \bar{\nu}_e(x) \nu_e(x) \\
&+ m_d \bar{d}(x) d(x) + m_u \bar{u}(x) u(x) \\
&- \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e A_\mu(x) j_{(em)}^\mu(x) \\
&- \frac{1}{2} \text{Tr} \mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu} + \text{Tr} J_{YM}^\mu A_\mu
\end{aligned} \tag{408}$$

where $m_{\nu_e} = \hbar\omega/c^2$, $m_u = \hbar\omega/c^2$, $m_d = \hbar\omega/c^2$, and $m_e = \hbar\omega/c^2$ are masses for neutrino, u-quark, d-quark and electron, respectively. The electric current is

$$\begin{aligned}
j_{(em)}^\mu(x) &= i\bar{e}(x)\gamma^\mu e(x) + i\frac{2}{3}\bar{u}(x)\gamma^\mu u(x) \\
&+ i\frac{1}{3}\bar{d}(x)\gamma^\mu d(x).
\end{aligned} \tag{409}$$

The $SU_{\text{Strong}}(3)$ color current for strong interaction is

$$J_{YM}^{a,\mu} = \bar{u}(x) i\gamma^\mu T^a u(x) + \bar{d}(x) i\gamma^\mu T^a d(x). \tag{410}$$

E. Knot physics for 3-level double-helix knot-crystal

From above discussion we have shown that after mapping to 2-level double-helix composite knot-crystal, the effective theory for a 2-level double-helix knot-crystal with $\Delta_{12} = 3.5$ is $SU(3) \otimes U(1)$ gauge theory with neutrino, electron, d-quark and u-quark. Such effective theory is similar to the Standard model. However, there are several differences: 1) the Standard model is $SU_{\text{Strong}}(3) \otimes SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge theory with Higgs mechanism due to spontaneous symmetry breaking; 2) the neutrino with tiny mass has only left-hand degrees of freedom; 3) there exists an $SU_{\text{weak}}(2)$ weak interaction for left-hand fermions[26]. To derive an exact formulation of SM model, we consider 3-level double-helix knot-crystal.

1. 3-level knot-crystal and standard knot-crystal

a. Standard knot-crystal For a 3-level double-helix knot-crystal, there exist 16 possible chiralities, i.e., $\mathbf{Z}_{L,L,LL}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{L,L,LR}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{L,L,RL}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{L,L,RR}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{L,R,LL}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{L,R,LR}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{L,R,RL}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{L,R,RR}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{R,L,LL}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{R,L,LR}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{R,L,RL}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{R,L,RR}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{R,R,LL}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{R,R,LR}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{R,R,RL}(\vec{e}, \vec{x}, t)$, $\mathbf{Z}_{R,R,RR}(\vec{e}, \vec{x}, t)$. Due to intrinsic symmetry between them, we take 3D 3-level knot-crystal with knot-function $\mathbf{Z}(\vec{e}, \vec{x}, t) = \mathbf{Z}_{L,L,LL}(\vec{e}, \vec{x}, t)$ as an example to learn knot physics for 3-level double-helix knot-crystal.

To show the hierarchy structure of a 3-level knot-crystal, we define the knot-functions to be

$$\mathbf{Z}(\vec{e}, \vec{x}, t) = \begin{pmatrix} z_{A,1}(\vec{x}, t) & z_{A,2}(\vec{x}, t) & z_{A,3}(\vec{x}, t) \\ z_{B,1}(\vec{x}, t) & z_{B,2}(\vec{x}, t) & z_{B,3}(\vec{x}, t) \end{pmatrix} \tag{411}$$

with

$$\begin{pmatrix} z_{A,3}(\vec{x}, t) \\ z_{B,3}(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} r_{A,3} e^{i\phi_{3,A}(\vec{x}, t)} \\ r_{B,3} e^{i\phi_{3,B}(\vec{x}, t)} \end{pmatrix}, \tag{412}$$

$$\begin{aligned}
\left(\frac{z_{A,3}(\vec{x}, t)}{r_{A,3}} \right)^{(\Delta_{23})_A} &= \frac{z_{A,2}(\vec{x}, t)}{r_{A,2}}, \\
\left(\frac{z_{B,3}(\vec{x}, t)}{r_{B,3}} \right)^{(\Delta_{23})_B} &= \frac{z_{B,2}(\vec{x}, t)}{r_{B,2}},
\end{aligned} \tag{413}$$

and

$$\begin{aligned} \left(\frac{z_{A,2}(\vec{x}, t)}{r_{A,2}} \right)^{(\Delta_{12})_A} &= \frac{z_{A,1}(\vec{x}, t)}{r_{A,1}}, \\ \left(\frac{z_{B,2}(\vec{x}, t)}{r_{B,2}} \right)^{(\Delta_{12})_B} &= \frac{z_{B,1}(\vec{x}, t)}{r_{B,1}}, \end{aligned} \quad (414)$$

where

$$\begin{aligned} \phi_{A,3}(\vec{x}, t) &= \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \omega_A t, \\ \phi_{B,3}(\vec{x}, t) &= \frac{\pi}{a}(\vec{e} \cdot \vec{x}) - \omega_B t. \end{aligned} \quad (415)$$

Here, the period ratios $(\Delta_{12})_{A/B}$ are the winding numbers of A/B brane between level-1 and level-2. The period ratios $(\Delta_{23})_{A/B}$ are the winding numbers of A/B brane between level-2 and level-3. $r_{A/B,1}$, $r_{A/B,2}$, $r_{A/B,3}$ are winding radii for level-1 knot-crystal, level-2 knot-crystal, level-3 knot-crystal, respectively. In general, we have $r_{A,3} = r_{B,3} = r_3$, $r_{A,2} = r_{B,2} = r_2$ and $r_{A,1} \simeq r_{B,1} = r_1$. a is a fixed length that denotes the half pitch of 3-th level windings and \vec{e} is winding direction in 3D space. The hierarchy recurrence relationship of 3-level double-helix knot-crystal is

$$\begin{aligned} \phi_{1,A}(\vec{x}, t) &= (\Delta_{12})_A \phi_{2,A}(\vec{x}, t) = (\Delta_{12})_A (\Delta_{23})_A \phi_{3,A}(\vec{x}, t), \\ \phi_{1,B}(\vec{x}, t) &= (\Delta_{12})_B \phi_{2,B}(\vec{x}, t) = (\Delta_{12})_B (\Delta_{23})_B \phi_{3,B}(\vec{x}, t). \end{aligned} \quad (416)$$

For 3-level double-helix knot-crystal, there exists generalized translation symmetry, i.e.,

$$\begin{aligned} \mathcal{T}(\Delta\vec{x}) \begin{pmatrix} z_{A,1}(\vec{x}, t) \\ z_{B,1}(\vec{x}, t) \end{pmatrix} &= \begin{pmatrix} \mathcal{T}(\Delta\vec{x}) z_{A,1}(\vec{x}, t) \\ \mathcal{T}(\Delta\vec{x}) z_{B,1}(\vec{x}, t) \end{pmatrix} \\ &= \begin{pmatrix} \exp[i\frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x}) (\Delta_{12})_A (\Delta_{23})_A] z_{A,1}(\vec{x}, t) \\ \exp[i\frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x}) (\Delta_{12})_B (\Delta_{23})_B] z_{B,1}(\vec{x}, t) \end{pmatrix}, \end{aligned} \quad (417)$$

$$\begin{aligned} \mathcal{T}(\Delta\vec{x}) \begin{pmatrix} z_{A,2}(\vec{x}, t) \\ z_{B,2}(\vec{x}, t) \end{pmatrix} &= \begin{pmatrix} \mathcal{T}(\Delta\vec{x}) z_{A,2}(\vec{x}, t) \\ \mathcal{T}(\Delta\vec{x}) z_{B,2}(\vec{x}, t) \end{pmatrix} \\ &= \begin{pmatrix} \exp[i\frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x}) (\Delta_{23})_A] z_{A,2}(\vec{x}, t) \\ \exp[i\frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x}) (\Delta_{23})_B] z_{B,2}(\vec{x}, t) \end{pmatrix}, \end{aligned} \quad (418)$$

and

$$\begin{aligned} \mathcal{T}(\Delta\vec{x}) \begin{pmatrix} z_{A,3}(\vec{x}, t) \\ z_{B,3}(\vec{x}, t) \end{pmatrix} &= \begin{pmatrix} \mathcal{T}(\Delta\vec{x}) z_{A,3}(\vec{x}, t) \\ \mathcal{T}(\Delta\vec{x}) z_{B,3}(\vec{x}, t) \end{pmatrix} \\ &= \begin{pmatrix} \exp[i\frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x})] z_{A,3}(\vec{x}, t) \\ \exp[i\frac{\pi}{a}(\vec{e} \cdot \Delta\vec{x})] z_{B,3}(\vec{x}, t) \end{pmatrix}. \end{aligned} \quad (419)$$

We point out that for the case of

$$\begin{aligned} \phi_{A,3}(\vec{x}, t) &= \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \omega_0 t, \\ \phi_{B,3}(\vec{x}, t) &= \frac{\pi}{a}(\vec{e} \cdot \vec{x}) - \omega_0 t, \\ (\Delta_{12})_A &= (\Delta_{12})_B = 3.5, \\ (\Delta_{23})_A &= (\Delta_{23})_B + \delta \gg |(\Delta_{12})_{A/B}|, \end{aligned} \quad (420)$$

the effective theory reproduces Standard model, an $SU_{\text{Strong}}(3) \otimes SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge theory with Higgs mechanism due to spontaneous symmetry breaking. δ is a very tiny number, $\delta \ll 1$ (for example, $\delta \sim 10^{-10}$). We call this special 3-level knot-crystal to be the standard knot-crystal. Fig.21 is an illustration of the standard knot-crystal.

The linear composite knots of standard knot-crystal are 3-th level windings that are composite objects of a 1-th level half winding with an extra 2-th-level-winding and 1-th level n_k -internal-winding, i.e.,

$$\begin{aligned} &\text{Composite knot} \\ &= \text{A 3-th level winding} \\ &+ \text{a 2-th level winding} \\ &+ \text{a bare knot (1-th half-winding)} \\ &+ n_k\text{-internal-winding.} \end{aligned} \quad (421)$$

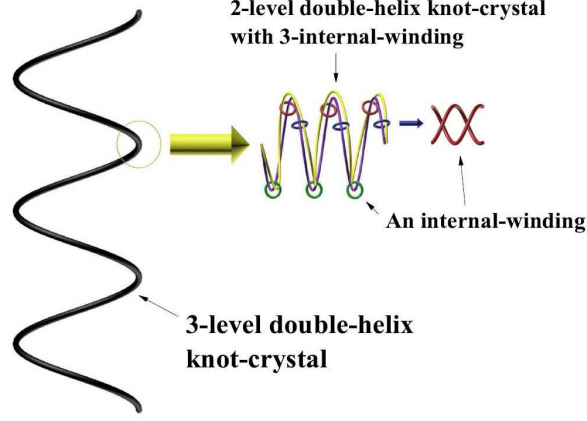


FIG. 21: An illustration of standard knot-crystal: a 3-level double-helix knot-crystal with $(\Delta_{12})_A = -(\Delta_{12})_B = 3 + \frac{1}{2}$, $(\Delta_{23})_A = (\Delta_{23})_B + \delta \gg (\Delta_{12})_{A/B}$. The large helix line is described by the knot-function of $z_{A/B,3}(\vec{x}, t)$, of which a point is a composite knot with 3-internal-winding. The small rings denote an internal-winding.

So, the classification of the linear composite knots of 3-level double-helix knot-crystal is same to the classification of those of 2-level double-helix knot-crystal with $(\Delta_{12})_{A/B} = 3.5$. Different composite knots have different internal windings $(\Delta_{12})_{A/B} = n_k$. There are 4 types of four-component (brane A/B and spin up/down) linear composite knots on double-helix knot-crystal, i.e., $n_k = 0, 1, 2, 3$.

A linear composite knot is a 3-th level winding along a given direction \vec{e} , of which the knot-function is given by

$$\mathbf{Z}_{\text{knot},A}(\vec{x}, t) = \begin{pmatrix} z_{A,1}(\vec{x}, t) & z_{A,2}(\vec{x}, t) & z_{A,3}(\vec{x}, t) \\ z_{A,1}(\vec{x}, t) & z_{A,2}(\vec{x}, t) & z_{A,3}(\vec{x}, t) \end{pmatrix} \quad (422)$$

where

$$\begin{aligned} z_{A,1}(\vec{x}, t) &= r_{A,1} \exp[\pm i \phi_{\text{knot},1}(\Delta \phi_{A,2}(\Delta \phi_{A,3}(\vec{x})), t)], \\ z_{A,2}(\vec{x}, t) &= r_{A,2} \exp[\pm i \phi_{\text{knot},2}(\Delta \phi_{A,3}(\vec{x}), t)], \\ z_{A,3}(\vec{x}, t) &= r_{A,3} \exp[\pm i \phi_{\text{knot},3}(\vec{x}, t)], \\ z_{B,1}(\vec{x}, t) &= r_{B,1}, \\ z_{B,2}(\vec{x}, t) &= r_{B,2}, \\ z_{B,3}(\vec{x}, t) &= r_{B,3}, \end{aligned} \quad (423)$$

and

$$\mathbf{Z}_{\text{knot},B}(\vec{x}, t) = \begin{pmatrix} z_{A,1}(\vec{x}, t) & z_{A,2}(\vec{x}, t) & z_{A,3}(\vec{x}, t) \\ z_{A,1}(\vec{x}, t) & z_{A,2}(\vec{x}, t) & z_{A,3}(\vec{x}, t) \end{pmatrix} \quad (424)$$

where

$$\begin{aligned} z_{A,1}(\vec{x}, t) &= r_{A,1}, \\ z_{A,2}(\vec{x}, t) &= r_{A,2}, \\ z_{A,3}(\vec{x}, t) &= r_{A,3}, \\ z_{B,1}(\vec{x}, t) &= r_{B,1} \exp[\pm i \phi_{\text{knot},1}(\Delta \phi_{B,2}(\Delta \phi_{B,3}(\vec{x})), t)], \\ z_{B,2}(\vec{x}, t) &= r_{B,2} \exp[\pm i \phi_{\text{knot},2}(\Delta \phi_{B,3}(\vec{x}), t)], \\ z_{B,3}(\vec{x}, t) &= r_{B,3} \exp[\pm i \phi_{\text{knot},3}(\vec{x}, t)]. \end{aligned} \quad (425)$$

Here we have

$$\phi_{\text{knot},3}(x'_1) = \begin{cases} 0, & x'_1 \in (-\infty, x_0] \\ -k_0(x'_1 - x_0), & x'_1 \in (x_0, x_0 + 2a] \\ 2\pi, & x'_1 \in (x_0 + 2a, \infty) \end{cases}, \quad (426)$$

$$\phi_{\text{knot},1}(\Delta\phi_{A/B,2}) = \left\{ \begin{array}{l} \phi_0, \Delta\phi_{A/B,3} \in (-\infty, -\pi] \\ \phi_0 + \Delta\phi_{A/B,3}, \Delta\phi_{A/B,3} \in (0, 2\pi] \\ \phi_0 + 2\pi, \Delta\phi_{A/B,3} \in (\pi, \infty) \end{array} \right\}, \quad (427)$$

and

$$\phi_{\text{knot},1}(\Delta\phi_{A/B,2}) = \left\{ \begin{array}{l} \phi_0 - \frac{\pi}{2}, \Delta\phi_{A/B,2} \in (-\infty, -\pi] \\ \phi_0 - \frac{\pi}{2} - \frac{(2n_k+1)}{2}\Delta\phi_{A/B,2}, \Delta\phi_{A/B,2} \in (0, 2\pi] \\ \phi_0 + \frac{\pi}{2}, \Delta\phi_{A/B,2} \in (\pi, \infty) \end{array} \right\}. \quad (428)$$

$x'_1 = \vec{x} \cdot \vec{e}$ is the coordination on the axis along a given direction \vec{e} and ϕ_0 is a constant angle, $k_0 = \frac{\pi}{a}$.

We may also generate a linear composite knot of standard knot-crystal by doing knot-operation by using similar approach on a 2-level double-helix knot-crystal.

b. Low energy effective model for standard knot-crystal For the quantum states of composite knot-crystal, there exists the continuous global translation symmetry

$$\begin{aligned} \hat{T}(\Delta\vec{x}) \left| \vec{K} \right\rangle &= e^{i\vec{k}_0 \cdot [2a[(\Delta\vec{x}) \bmod 2a]]} \\ &\times e^{i\vec{K} \cdot [(\Delta\vec{x}) - 2a[(\Delta\vec{x}) \bmod 2a]]} \left| \vec{K} \right\rangle \end{aligned} \quad (429)$$

where $[(\Delta\vec{x}) - 2a[(\Delta\vec{x}) \bmod 2a]]/2a = \vec{X}$ denotes an integer 3-th level winding number along the direction $\vec{k}_0 = \frac{\pi}{a}\vec{e}$. For the case of $(\Delta\vec{x}) \bmod 2a = 0$, we have

$$\hat{T}(\Delta\vec{x}) \left| \vec{p}_X \right\rangle = e^{i\vec{p}_X \cdot (\Delta\vec{x})/\hbar} \left| \vec{p}_X \right\rangle. \quad (430)$$

We define the energy cost from brane-curved between two nearest-neighbor composite knots (that is $(\Delta_{12})_{A/B} (\Delta_{23})_{A/B}$ internal-windings) to be J and the energy cost from balancedly-rotating of Dirac knot-crystal to be $\hbar\omega$. For the 3-level knot-crystal, the effective Hamiltonian of a composite knot is obtained as

$$\hat{\mathcal{H}}_{3D} = \int (\psi^\dagger \hat{h}_{3D} \psi) d^3X \quad (431)$$

where

$$\hat{h}_{3D} = \vec{\Gamma} \cdot \vec{p}_X + m\Gamma^5 \quad (432)$$

and $\vec{p}_X = \frac{1}{2a}i\frac{d}{dX}$. Here, a universal light speed for different types linear composite knots $c \equiv aJ$ is set to be unit. In addition, the Planck constant is a universal value for different types of composite knots as[27]

$$\begin{aligned} \hbar &= \frac{1}{(\Delta_{12})_{A/B} (\Delta_{23})_{A/B}} \frac{a \cdot k_B T_{\text{brane}}}{3 \cdot \pi \cdot c_{5D}} \\ &= \frac{a \cdot k_B T_{\text{brane}}}{3 \cdot (\Delta_{12})_{A/B} (\Delta_{23})_{A/B} \pi \cdot c_{5D}}. \end{aligned} \quad (433)$$

c. Chiral property for standard knot-crystal Before discussing the chiral effect from mismatch between two branes $(\Delta_{23})_A - (\Delta_{23})_B = \delta$, we show the effect of a neutrino on a symmetric knot-crystal that is described by

$$\begin{aligned} \phi_{3,A}(\vec{x}, t) &= \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \omega_0 t, \\ \phi_{3,B}(\vec{x}, t) &= \frac{\pi}{a}(\vec{e} \cdot \vec{x}) - \omega_0 t, \\ (\Delta_{12})_A &= (\Delta_{12})_B = 3.5, \\ (\Delta_{23})_A &= (\Delta_{23})_B \gg |(\Delta_{12})_{A/B}|, \end{aligned} \quad (434)$$

where $(\Delta_{23})_A$ is an integer number and a is a fixed length that denotes the half pitch of 3-th windings.

In the low energy limit, $E \ll \frac{c\hbar}{a}\delta$, the wave-function of a composite knot has uniform distribution on $N = \delta^{-1}$ big-rings for the symmetric knot-crystal. We assume the ground state of the symmetric knot-crystal to be $|\text{vac}\rangle_{\text{symmetric}}$. Then, the excitation state of a single neutrino is

$$v_{e,R}^\dagger(\vec{p}_X) |\text{vac}\rangle_{\text{symmetric}} = |\vec{p}_X\rangle_{v_e,R} \quad (435)$$

where \vec{p}_X is the momentum. For the state $|X_0\rangle_{v_e,R}$, a right-band neutrino for the symmetric knot-crystal is just a total mismatch between two branes $(\Delta_{23})_A(X_0) - (\Delta_{23})_B(X_0) = \frac{1}{2}$.

Now we study the chiral property of the composite knot of standard knot-crystal that is described by

$$\begin{aligned}\phi_{3,A}(\vec{x}, t) &= \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \omega_0 t, \\ \phi_{3,B}(\vec{x}, t) &= \frac{\pi}{a}(\vec{e} \cdot \vec{x}) - \omega_0 t, \\ (\Delta_{12})_A &= (\Delta_{12})_B = 3.5, \\ (\Delta_{23})_A &= (\Delta_{23})_B + \delta \gg |(\Delta_{12})_{A/B}|.\end{aligned}\tag{436}$$

From above discussion, the total local mismatch between two branes $(\Delta_{23})_A(X_0) - (\Delta_{23})_B(X_0) = \frac{1}{2}$ can be regarded as a right-hand neutrino. So, a uniform distribution of mismatch between two branes $(\Delta_{23})_A - (\Delta_{23})_B = \delta$ becomes a uniform distribution of right-hand neutrinos with a probability δ on each big-ring. Now the ground state of standard knot-crystal is denoted by $|\text{vac}\rangle_{\text{standard}}$. The relationship between $|\text{vac}\rangle_{\text{standard}}$ and $|\text{vac}\rangle_{\text{symmetric}}$ is

$$\begin{aligned}|\text{vac}\rangle_{\text{standard}} &= \prod_X v_{e,R}^\dagger(\vec{X}) |\text{vac}\rangle_{\text{symmetric}} \\ &= \prod_{\vec{p}_X} v_{e,R}^\dagger(\vec{p}_X) |\text{vac}\rangle_{\text{symmetric}}.\end{aligned}\tag{437}$$

We may regard that δ^{-1} big-rings is filled by a right-hand neutrino. The ground state of $|\text{vac}\rangle_{\text{standard}}$ becomes a semi-metal state for right-hand neutrino.

In low energy limit $E \ll \frac{c\hbar\delta}{a}$, since right-hand neutrinos have been filled on the ground state, one cannot generate another right-hand neutrino anymore[28]. As a result, only left-hand neutrino can be generated or right-hand neutrino can be annihilated as

$$\begin{aligned}|X\rangle_{v_e,L} &= v_{e,L}^\dagger(\vec{X}_0) |\text{vac}\rangle_{\text{standard}} \\ &= \prod_{X \neq X_0} v_{e,R}^\dagger(\vec{X}) |\text{vac}\rangle_{\text{symmetric}}.\end{aligned}\tag{438}$$

This is the origin of chirality for neutrinos and weak interaction, i.e.,

Chirality \rightarrow Intrinsic uniform mismatch between two branes.

As a result, in low energy limit $E \ll \frac{c\hbar}{a}\delta$, the effective Lagrangian of elementary particles becomes

$$\begin{aligned}\mathcal{L}_{\text{fermion}}(x) &= \bar{\nu}_{e,L}(\vec{X}) i\gamma^\mu \partial_\mu \nu_{e,L}(\vec{X}) + \bar{e}(\vec{X}) i\gamma^\mu \partial_\mu e(\vec{X}) \\ &\quad + \bar{d}(\vec{X}) i\gamma^\mu \partial_\mu d(\vec{X}) + \bar{u}(\vec{X}) i\gamma^\mu \partial_\mu u(\vec{X}) \\ &\quad + m_e \bar{e}(\vec{X}) e(\vec{X}) + m_d \bar{d}(\vec{X}) d(\vec{X}) + m_u \bar{u}(\vec{X}) u(\vec{X}).\end{aligned}\tag{439}$$

There is no mass term for the left-hand neutrinos.

2. Electro-weak $\text{SU}_{\text{weak}}(2) \otimes \text{U}_Y(1)$ gauge symmetry for standard knot-crystal

An $\text{SU}_{\text{weak}}(2) \otimes \text{U}_Y(1)$ gauge field that couples neutrino and electron characterizes the interaction by exchanging the quantum fluctuations of internal-windings in big-rings. The quantum fluctuations of $\text{SU}_{\text{weak}}(2)$ gauge theory come from the density-fluctuations of internal-windings on a big-ring and the quantum fluctuations of $\text{U}_Y(1)$ gauge theory come from the position-fluctuations of composite knots and internal-windings on a big-ring.

a. $\text{SU}_{\text{weak}}(2)$ weak gauge symmetry for leptons In big-ring limit, the number of internal-windings on a big-ring $n_{\text{big-ring}} = (\Delta_{12})_A (\Delta_{23})_A$ is very large, $n_{\text{big-ring}} \gg 1$. A neutrino is really bare half-winding with $n_{\nu_e} = (\Delta_{12})_{A/B} = 0$ and an electron is a half-winding with 3-internal-windings $n_e = (\Delta_{12})_{A/B} = 3$. When the internal-windings become uniform distributed in a big-ring, the total number of internal-windings on a big-ring for a uniform distributed neutrino and that for a uniform distributed electron becomes $n_{\nu_e} + n_{\text{big-ring}}$ and $n_e + n_{\text{big-ring}}$, respectively. So, the density-changing of internal-windings induced by a uniform distributed neutrino on a ring is

$$\Delta\rho_{\text{internal-windings},\nu_e} = \frac{n_{\nu_e}}{n_{\text{big-ring}}} = 0\tag{440}$$

and the density-changing of internal-windings induced by a uniform distributed electron on a ring is

$$\Delta\rho_{\text{internal-windings},e} = \frac{n_e}{n_{\text{big-ring}}} = \frac{3}{n_{\text{big-ring}}}. \quad (441)$$

According the big-ring condition $n_{\text{big-ring}} \gg 1$, we have

$$\rho_{\text{internal-windings},\nu_e} \simeq \rho_{\text{internal-windings},e}. \quad (442)$$

That means without considering the charge degree of freedom, we cannot distinguish a uniform distributed left-hand electron from a uniform distributed left-hand neutrino on a big-ring. Therefore, we may regard a uniform distributed left-hand neutrino ν_e and a uniform distributed left-hand electron e to be same object of an $\text{SU}_{\text{weak}}(2)$ gauge symmetry on a big-ring. The quantum fluctuations of internal-windings on the big-ring play the role of the non-Abelian gauge fields for the $\text{SU}_{\text{weak}}(2)$ gauge symmetry.

We then consider a left-hand electron e to be a composite knot with

$$n_e = (n_e)_{\text{SU}_{\text{weak}}(2)} + (n_e)_{\text{U}_Y(1)} = 3$$

where $(n_e)_{\text{SU}_{\text{weak}}(2)} = \frac{3}{2}$ and $(n_e)_{\text{U}_Y(1)} = \frac{3}{2}$ and a left-hand neutrino ν_e to be a composite knot with

$$n_{\nu_e} = (n_{\nu_e})_{\text{SU}_{\text{weak}}(2)} + (n_{\nu_e})_{\text{U}_Y(1)} = 0$$

where $(n_{\nu_e})_{\text{SU}_{\text{weak}}(2)} = -\frac{3}{2}$ and $(n_{\nu_e})_{\text{U}_Y(1)} = \frac{3}{2}$.

To characterize the property of opposite-component of internal-windings for leptons

$$(n_e)_{\text{SU}_{\text{weak}}(2)} = -(n_{\nu_e})_{\text{SU}_{\text{weak}}(2)} = \frac{3}{2},$$

a left-hand neutrino ν_e and a left-hand electron e make up a lepton $\text{SU}_{\text{weak}}(2)$ spinor

$$\psi_{\text{Lepton}} = \begin{pmatrix} \nu_e \\ e \end{pmatrix}. \quad (443)$$

These considerations lead us to assign the left-handed components of the lepton fields to doublets of $\text{SU}_{\text{weak}}(2)$

$$\psi_{\text{Lepton},L} = \frac{1}{2}(1 + \gamma_5) \begin{pmatrix} \nu_e \\ e \end{pmatrix} \quad (444)$$

The right-handed components are assigned to singlets of $\text{SU}_{\text{weak}}(2)$ that has no neutrino and we have

$$\psi_{\text{Lepton},R} = e_R = \frac{1}{2}(1 - \gamma_5)e. \quad (445)$$

As a result, we have

$$\begin{aligned} \psi_{\text{Lepton},L} &\rightarrow e^{i\vec{\tau}\vec{\theta}(\vec{X})}\psi_{\text{Lepton},L}, \\ \psi_{\text{Lepton},R} &\rightarrow \psi_{\text{Lepton},R} \end{aligned} \quad (446)$$

with $\vec{\tau}$ the three Pauli matrices.

b. $\text{U}_Y(1)$ supercharge gauge symmetry for leptons To characterize the property of identical-component of internal-windings for leptons

$$(n_e)_{\text{U}_Y(1)} = (n_{\nu_e})_{\text{U}_Y(1)} = \frac{3}{2},$$

a left-hand neutrino ν_e and a left-hand electron e have an additional charge degree of freedom from $\text{U}_Y(1)$ gauge symmetry. Such charge degree of freedom is called the supercharge degrees of freedom. The supercharge \mathcal{Y} of left-hand electron and left-hand neutrino with internal-windings $(n_e)_{\text{U}_Y(1)} = (n_{\nu_e})_{\text{U}_Y(1)} = \frac{3}{2}$ is

$$\mathcal{Y}(\psi_{\text{Lepton},L}) = -1. \quad (447)$$

A unit internal-winding has supercharge \mathcal{Y} to be $-\frac{2}{3}$.

For a right-hand electron, there are 3-internal-windings. The 3-internal-windings have supercharge \mathcal{Y} to be $3 \times (-\frac{2}{3}) = -2$. Therefore, right-hand electrons e_R have -2 supercharge as

$$\mathcal{Y}(\psi_{\text{Lepton},R}) = -2. \quad (448)$$

c. Electro-weak $SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge symmetry for quarks In addition, we consider the electro-weak interaction for quarks. In big-ring limit, a d-quark has 1-internal-winding $n_{\text{d-quark}} = (\Delta_{12})_{\text{A/B}} = 1$ and an anti-u-quark has 2-internal-winding $n_{\text{u}^*-\text{quark}} = (\Delta_{12})_{\text{A/B}} = -2$. When the internal-windings become uniform distributed in a big-ring, the total number of internal-windings for a uniform distributed d-quark on a big-ring becomes $n_{\text{d-quark}} + (\Delta_{12})_{\text{A}} (\Delta_{23})_{\text{A}}$ and the total number of internal-windings for a uniform distributed anti-u-quark on a ring becomes $n_{\text{u}^*-\text{quark}} + n_{\text{big-ring}}$. According the big-ring condition $n_{\text{big-ring}} \gg 1$, we have

$$n_{\text{d-quark}} + n_{\text{big-ring}} \simeq n_{\text{u}^*-\text{quark}} + n_{\text{big-ring}}.$$

We may also regard a uniform distributed left-hand d-quark and a uniform distributed left-hand anti-u-quark to be same object of an $SU_{\text{weak}}(2)$ gauge symmetry on the big-ring.

We consider a left-hand d-quark to be a composite knot as

$$n_{\text{d-quark}} = (n_{\text{d-quark}})_{SU_{\text{weak}}(2)} + (n_{\text{d-quark}})_{U_Y(1)}$$

where $(n_{\text{d-quark}})_{SU_{\text{weak}}(2)} = \frac{3}{2}$ and $(n_{\text{d-quark}})_{U_Y(1)} = -\frac{1}{2}$ and a left-hand u-quark to be a knot with 1.5-internal-winding,

$$n_{\text{u}^*-\text{quark}} = (n_{\text{u}^*-\text{quark}})_{SU_{\text{weak}}(2)} + (n_{\text{u}^*-\text{quark}})_{U_Y(1)}$$

where $(n_{\text{u-quark}})_{SU_{\text{weak}}(2)} = -\frac{3}{2}$ and $(n_{\text{u-quark}})_{U_Y(1)} = -\frac{1}{2}$.

To characterize the property of opposite-component of internal-windings for \bar{u} and d as

$$(n_{\text{d-quark}})_{SU_{\text{weak}}(2)} = -(n_{\text{u-quark}})_{SU_{\text{weak}}(2)} = \frac{3}{2},$$

\bar{u} and d make up a quark $SU_{\text{weak}}(2)$ spinor[29]

$$\psi_{\text{quark}} = \begin{pmatrix} \bar{u} \\ d \end{pmatrix}. \quad (449)$$

They are fractionally charged and come each in three ‘‘colours’’,

$$\psi_{\text{quark},L} = \frac{1}{2}(1 + \gamma_5) \begin{pmatrix} \bar{u} \\ d \end{pmatrix}. \quad (450)$$

On the other hand, to characterize the property of identical-component of internal-windings for \bar{u} and d as

$$(n_{\text{d-quark}})_{SU_{\text{weak}}(2)} = (n_{\text{u-quark}})_{SU_{\text{weak}}(2)} = -\frac{1}{2},$$

a left-hand d-quark and a left-hand anti-u-quark have the supercharge \mathcal{Y} to be

$$\mathcal{Y}(\psi_{\text{quark},L}) = -\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}. \quad (451)$$

Since a unit internal-winding has supercharge \mathcal{Y} to be $-\frac{2}{3}$, we find that the super charge for quarks u_R , d_R to be

$$\mathcal{Y}(u_R) = -\frac{4}{3}, \quad \mathcal{Y}(d_R) = -\frac{2}{3}. \quad (452)$$

d. Effective Lagrangian of electro-weak $SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge theory Finally, we write down the effective Lagrangian of electro-weak $SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge theory

$$\begin{aligned} \mathcal{L} \Big(\Big| (\Delta_{12})_{\text{A/B}} \Big| = 3 + \frac{1}{2}, (\Delta_{23})_{\text{A}} = (\Delta_{23})_{\text{B}} + 1/2 \gg (\Delta_{12})_{\text{A/B}} \Big) \\ = \mathcal{L}_{\text{fermion}} + \mathcal{L}_Y(U_Y(1)) + \mathcal{L}_{\text{weak}}(SU_{\text{weak}}(2)) \\ = \text{Tr} \bar{\psi}_L i \gamma^\mu (\partial_\mu - i g W_\mu + i \frac{g'}{2} B_\mu) \psi_L \\ + \bar{\psi}_R i \gamma^\mu (\partial_\mu + i g' B_\mu) \psi_R \\ - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \text{Tr} \frac{1}{2} W_{\mu\nu} W^{\mu\nu} \end{aligned} \quad (453)$$

where W_μ and B_μ denote the gauge fields associated to weak $SU_{\text{weak}}(2)$ and super-charge $U_Y(1)$ respectively, of which the corresponding field strengths are $W_{\mu\nu}$ and $B_{\mu\nu}$. The two coupling constants g and g' correspond to the groups $SU_{\text{weak}}(2)$ and $U_Y(1)$ respectively. Because neutrino has only left-hand degrees of freedom, the charged W 's couple only to the left-handed components of the lepton fields.

In particular, we point out that electro-weak $SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge symmetry is an effective gauge symmetry as a physical consequence of big-ring approximation. So, electro-weak $SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge symmetry comes from the redundancies of 2-level composite linear knots in big-rings. We may split the standard knot-crystal into a 3-level Dirac knot-crystal $\mathbf{Z}_{\text{Dirac}}(\vec{e}, \vec{x}, t)$ and a 3-level substrate knot-crystal $\mathbf{Z}_{\text{substrate}}(\vec{e}, \vec{x}, t)$ and develop $U_Y(1)$ theory from continuous internal translation symmetry for composite knot-crystal in big-ring limit. However, $SU_{\text{weak}}(2)$ gauge theory don't come from a $n = 2$ discrete internal translation symmetry. It is due to the irrelevant of 3-internal-winding in a big-ring with a lot of internal-windings. As a result, the quantum fluctuations of $SU_{\text{weak}}(2)$ gauge theory comes from the density-fluctuations of internal-windings on a big-ring and the quantum fluctuations of $U_Y(1)$ gauge theory come from the position-fluctuations of composite knots on a big-ring.

3. Higgs mechanism and spontaneous symmetry breaking for standard knot-crystal

In this part we focus on a balancedly-rotating standard knot-crystal, of which the knot-functions become time-dependent

$$z_{1,A}(\vec{X}) \rightarrow z_{1,A}(\vec{X}, t) = z_{1,A}(\vec{X}) \exp(i\omega_A t) \quad (454)$$

and

$$z_{1,B}(\vec{X}) \rightarrow z_{1,B}(\vec{X}, t) = z_{1,B}(\vec{X}) \exp(-i\omega_B t) \quad (455)$$

The mass of different types fermions comes from brane-rotation, i.e., $\omega_A = -\omega_B = \omega_0$.

a. Rotating field as Higgs field It is pointed out that the fluctuating rotating velocity of a standard knot-crystal $\omega_0 \rightarrow \omega(\vec{X}, t)$ plays the role of Higgs field $\Phi(\vec{X}, t)$ in SM[30]. And the condensation of Higgs field corresponds to a finite rotating velocity of knot-crystal. It is the rotating field that gives masses to knots (electrons and quarks) and three vector gauge bosons. Due to chirality, left-hand neutrino doesn't obtain mass.

We then study the properties of rotating field $\omega(\vec{X}, t)$ (that is really the Higgs field $\Phi(\vec{X}, t)$ in SM). It is known for the rotating brane, we have the knot's wave-function to be

$$\psi'_i(t) = e^{i\tau_x \omega(\vec{X}, t) \cdot t} \cdot \psi_i. \quad (456)$$

Consequently, the effect of rotating field is to change $\psi_L(\vec{X})$ to $\psi_R(\vec{X})$ and there appears an extra term in Hamiltonian as

$$\sum_i (\psi_i^\dagger \omega_i \tau_x \psi_i) = \sum_i (\psi_{i,R}^\dagger \omega_i \psi_{i,L}). \quad (457)$$

Because of

$$\mathcal{Y}(\psi_{\text{Lepton},L}) = -1, \quad \mathcal{Y}(\psi_{\text{Lepton},R}) = 0, \quad (458)$$

the $U_Y(1)$ super-charge of $\omega(\vec{X}, t)$ must be

$$\begin{aligned} \mathcal{Y}(\omega(\vec{X}, t)) &= \mathcal{Y}(\psi_{\text{Lepton},R}) - \mathcal{Y}(\psi_{\text{Lepton},L}) \\ &= 1. \end{aligned} \quad (459)$$

On the other hand, due to

$$\begin{aligned} \psi_L(\vec{X}) &\rightarrow e^{i\vec{\tau}\vec{\theta}(\vec{X})} \psi(\vec{X}), \\ \psi_R(\vec{X}) &\rightarrow \psi_R(\vec{X}), \end{aligned} \quad (460)$$

$\omega(\vec{X}, t)$ must be an $SU_{\text{weak}}(2)$ complex doublet as

$$\begin{aligned} \omega(\vec{X}, t) &= \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \\ \omega(\vec{X}, t) &\rightarrow e^{i\vec{\tau}\vec{\theta}(X)} \omega(\vec{X}, t). \end{aligned} \quad (461)$$

Next, we write down the Lagrangian of the rotating field $\omega(\vec{X}, t)$. Because the rotating field $\omega(\vec{X}, t)$ is an $\text{SU}_{\text{weak}}(2)$ complex doublet and has supercharge $\mathcal{Y} = 1$, we get the kinetic term of rotating field $\omega(\vec{X}, t)$ that is

$$|(\partial_\mu - ig\frac{\vec{\tau}}{2} \cdot \vec{W}_\mu - ig'\frac{B_\mu}{2})\omega(\vec{X}, t)|^2. \quad (462)$$

To obtain the finite rotating velocity, we add a phenomenological term

$$V(\omega(\vec{X}, t)) = \mu^2 \omega^\dagger(\vec{X}, t) \omega(\vec{X}, t) + \lambda \left| \omega(\vec{X}, t)^\dagger \omega(\vec{X}, t) \right|^2. \quad (463)$$

Finally, by adding Yukawa coupling between the rotating field and fermions, the full Lagrangian of rotating field $\omega(\vec{X}, t)$ is given by

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} = & |(\partial_\mu - ig\frac{\vec{\tau}}{2} \cdot \vec{W}_\mu - ig'\frac{B_\mu}{2})\omega(\vec{X}, t)|^2 \\ & - V(\omega(\vec{X}, t)) + \bar{\psi}_{\text{Lepton}, L} G_{\text{Lepton}} \omega(\vec{X}, t) \bar{\psi}_{\text{Lepton}, R} \\ & + \bar{\psi}_{\text{quark}, L} G_{\text{quark}} \omega(\vec{X}, t) \psi_{\text{quark}, R} + h.c. \end{aligned} \quad (464)$$

where the unknown coupling constant matrices G_{Lepton} and G_{quark} will determine the masses of particles.

b. Spontaneous symmetry breaking A finite velocity is given by minimizing $\omega(\vec{X}, t)$, of which the expected value is $\omega_0^2 = -\mu^2/\lambda$. Then, the weak gauge symmetry is spontaneously broken, we get a finite rotating velocity:

$$\langle \omega(\vec{X}, t) \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \omega_0 \end{pmatrix} + \delta\omega(\vec{X}, t). \quad (465)$$

A finite rotating velocity creates a mass term for the charged leptons, leaving the neutrinos massless,

$$m_e = \frac{1}{\sqrt{2}} G_e \omega_0, \quad m_{\nu_e} = 0. \quad (466)$$

For the system with finite rotating velocity, we produce masses for the quarks given by $m_u = G_u \omega_0$, $m_d = G_d \omega_0$. Because there is no right-hand neutrino, the mass of neutrino is zero, $m_{\nu_e} \equiv 0$. In addition, the rotating field also has mass that is

$$m_{\text{Higgs}} = \sqrt{-2\mu^2} = \sqrt{2\lambda\omega_0^2}. \quad (467)$$

c. Higgs mechanism The finite rotating velocity plays the role of Higgs condensation. Under Higgs condensation, there exists Higgs mechanism. The Higgs mechanism breaks the original gauge symmetry according to $\text{SU}_{\text{weak}}(2) \otimes \text{U}_Y(1) \rightarrow \text{U}_{\text{EM}}(1)$.

As a result, the $\text{SU}_{\text{weak}}(2)$ gauge fields obtain masses from the following terms[31]

$$\frac{1}{8} \omega_0^2 [g^2 (W_\mu^1 W^{1\mu} + W_\mu^2 W^{2\mu}) + (g' B_\mu - g W_\mu^3)^2]. \quad (468)$$

The mass for the charged vector bosons $W_\mu^\pm = (W_\mu^1 \mp iW_\mu^2)/\sqrt{2}$ is

$$m_W = \frac{\omega_0 g}{2}. \quad (469)$$

After diagonalisation, the gauge fields B_μ and W_μ^3 are transformed into gauge fields Z_μ and A_μ from the following relations

$$\begin{aligned} Z_\mu &= \cos \theta_W B_\mu - \sin \theta_W W_\mu^3, \\ A_\mu &= \cos \theta_W B_\mu + \sin \theta_W W_\mu^3, \end{aligned} \quad (470)$$

with $\tan \theta_W = g'/g$, of which the masses are

$$\begin{aligned} m_Z &= \frac{\omega_0 (g^2 + g'^2)^{1/2}}{2} = \frac{m_W}{\cos \theta_W}, \\ m_A &= 0. \end{aligned} \quad (471)$$

The “Weinberg angle” θ_W becomes the angle between the original $U(1)$ and the one left unbroken.

The neutral gauge bosons A_μ are massless and will be identified with the photons. Now, the gauge symmetry $U_{EM}(1)$ accompanying A_μ is to change the position of the internal-windings of the composite knots that will never be broken. That is

$$\begin{aligned}\psi(\vec{X}) &= \begin{pmatrix} \psi_L(\vec{X}) \\ \psi_R(\vec{X}) \end{pmatrix} \rightarrow \begin{pmatrix} \psi_L(x)e^{-ie\phi(\vec{X})} \\ \psi_R(x)e^{-ie\phi(\vec{X})} \end{pmatrix} \\ &\rightarrow e^{-ie\phi(\vec{X})}\psi(\vec{X}).\end{aligned}\tag{472}$$

Then, the electric charge operator e will be a linear combination of T_3 and \mathcal{Y} as

$$e = T_3 + \frac{\mathcal{Y}}{2}.\tag{473}$$

In summary, we have

Rotating velocity $\omega(\vec{X}, t) \rightarrow$ Higgs field.

F. GUT as effective theory of standard knot-crystal

In the last section, we show the final result. Our universe is a 3D 3-level double-helix knot-crystal ($\mathcal{N} = 2$) with $(\Delta_{12})_A = (\Delta_{12})_B = 3.5$, $(\Delta_{23})_A = (\Delta_{23})_B + \delta \gg (\Delta_{12})_{A/B}$. We call it standard knot-crystal, of which the hierarchy series is

$$\begin{pmatrix} \{3.5, N\} \\ \{3.5, N + \delta\} \end{pmatrix}\tag{474}$$

where N is a very large number, $N \gg 1$ (for example, $N \sim 10^{15}$) and δ is a very tiny number, $\delta \ll 1$ (for example, $\delta \sim 10^{-10}$). For this reason, the hierarchy series is really a code of our universe. We call it *hierarchy series of universe*. The low energy effective theory is just the Standard model – an $SU_{\text{Strong}}(3) \otimes SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge theory with Higgs mechanism due to spontaneous symmetry breaking.

Before considering Higgs condensation or brane-rotating $\omega_0 = 0$, the low energy effective Lagrangian density ($E \ll \frac{c\hbar}{a}\delta$) is

$$\begin{aligned}\mathcal{L}_{\text{SM}} &= \mathcal{L}_{\text{fermion}} + \mathcal{L}_Y(U_Y(1)) + \mathcal{L}_{\text{strong}}(SU_{\text{Strong}}(3)) \\ &\quad + \mathcal{L}_{\text{weak}}(SU_{\text{weak}}(2)) + \mathcal{L}_{\text{Higgs}}\end{aligned}\tag{475}$$

where

$$\begin{aligned}\mathcal{L}_{\text{fermion}} &= \text{Tr} \bar{\psi}_L i\gamma^\mu (\partial_\mu - ig\frac{\vec{\tau}}{2} \cdot \vec{W}_\mu + ig'B_\mu) \psi_L \\ &\quad + \bar{\psi}_R i\gamma^\mu (\partial_\mu + ig'B_\mu) \psi_R \\ &\quad + \text{Tr} \bar{\psi}_{\text{quark},L} i(\partial_\mu - ig\frac{\vec{\tau}}{2} \cdot \vec{W}_\mu - ig'B_\mu) \psi_{\text{quark},L} \\ &\quad + \bar{u}_R i(\partial_\mu + i\frac{2g'}{3}B_\mu) u_R + \bar{d}_R i(\partial_\mu + i\frac{g'}{3}B_\mu) d_R,\end{aligned}\tag{476}$$

$$\mathcal{L}_Y(U_Y(1)) = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu},\tag{477}$$

$$\mathcal{L}_{\text{strong}}(SU_{\text{Strong}}(3)) = -\frac{1}{2}\text{Tr}\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu}\tag{478}$$

$$\mathcal{L}_{\text{weak}}(SU_{\text{weak}}(2)) = -\text{Tr}\frac{1}{2}W_{\mu\nu}W^{\mu\nu},\tag{479}$$

$$\begin{aligned}
\mathcal{L}_{\text{Higgs}} = & |(\partial_\mu - ig\frac{\vec{\tau}}{2} \cdot \vec{W}_\mu - i\frac{g'}{2}B_\mu)\omega|^2 \\
& - V(\omega) + \bar{\psi}_{\text{quark},L}G_{\text{quark}}\omega\psi_{\text{quark},R} \\
& + \bar{\psi}_{\text{Lepton},L}G_{\text{Lepton}}\omega\psi_{\text{Lepton},R} + h.c..
\end{aligned} \tag{480}$$

After considering the Higgs condensation $\omega_0 \neq 0$, we have the low energy effective Lagrangian as

$$\begin{aligned}
\mathcal{L}_{\text{SM}} = & \mathcal{L}_{\text{fermion}} + \mathcal{L}(\text{U}_{\text{EM}}(1)) + \mathcal{L}_{\text{weak}}(\text{SU}_{\text{weak}}(2)) \\
& + \mathcal{L}_{\text{strong}}(\text{SU}_{\text{Strong}}(3)) + \mathcal{L}_{\text{Higgs}} \\
= & \bar{\nu}_{L,e}i\gamma^\mu\partial_\mu\nu_{L,e} + \bar{e}_Ri\gamma^\mu\partial_\mu e \\
& + \bar{u}i\gamma^\mu\partial_\mu u + \bar{d}i\gamma^\mu\partial_\mu d + m_e\bar{e}e \\
& + m_d\bar{d}d + m_u\bar{u}u. \\
& - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \text{Tr}\frac{1}{2}W_{\mu\nu}W^{\mu\nu} - \frac{1}{2}\text{Tr}\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu} \\
& + eA_\mu(x)j_{(em)}^\mu + \text{Tr}J_{\text{YM}}^\mu\mathcal{A}_\mu \\
& + \frac{1}{8}\omega_0^2[g^2(W_\mu^1W^{1\mu} + W_\mu^2W^{2\mu}) + (g'B_\mu - gW_\mu^3)^2] \\
& + g_w(W_\mu^+j_w^\mu - + W_\mu^-j_w^\mu) + g_zW_\mu^zj_w^\mu \\
& + |\partial_\mu\omega|^2 + m_{\text{Higgs}}|\omega|^2 + \dots
\end{aligned} \tag{481}$$

where the electric current is

$$\begin{aligned}
j_{(em)}^\mu = & i\bar{e}\gamma^\mu e + i\frac{2}{3}\bar{u}\gamma^\mu u \\
& + i\frac{1}{3}\bar{d}\gamma^\mu d,
\end{aligned} \tag{482}$$

the weak current is

$$\begin{aligned}
j_{w-}^\mu = & i\bar{e}\gamma_\mu\nu_e + iu^T\gamma_\mu d, \\
j_{w+}^\mu = & i\bar{\nu}_e\gamma_\mu e + id^T\gamma_\mu u,
\end{aligned} \tag{483}$$

and the color current is

$$J_{\text{YM}}^{a,\mu} = \bar{u}i\gamma^\mu T^a u + \bar{d}i\gamma^\mu T^a d. \tag{484}$$

This is exact one-flavor SM, an $\text{SU}_{\text{Strong}}(3) \otimes (\text{SU}(2))_{\text{weak}} \otimes \text{U}_Y(1)$ gauge theory with Higgs mechanism due to spontaneous symmetry breaking.

From above discussion, we may slightly *tune* knot-crystal and get a new quantum field theory that is different to standard knot-crystal.

The first example is a 3D 3-level double-helix knot-crystal with the hierarchy series

$$\left(\begin{array}{c} \{3.5, N + \delta\} \\ \{3.5, N + \delta\} \end{array} \right) \tag{485}$$

where N is a large integer number. The low energy effective theory becomes an $\text{SU}_{\text{Strong}}(3) \otimes \text{SU}_{\text{weak}}(2) \otimes \text{U}_Y(1)$ gauge theory without Higgs mechanism. There exist massless neutrinos with both left-hand and left-hand degrees of freedom. And we have the $\text{SU}_{\text{weak}}(2)$ weak interaction for both left-hand and right-hand fermions. Without Higgs mechanism, the $\text{SU}_{\text{weak}}(2) \otimes \text{U}_Y(1)$ gauge symmetry is not broken. As a result, there is no mass term for gauge fields.

Another example is a 3D 3-level double-helix knot-crystal with the hierarchy series

$$\left(\begin{array}{c} \{3.5, N\} \\ \{3.5, N\} \end{array} \right) \tag{486}$$

where N is a large integer number. The low energy effective theory is also an $\text{SU}_{\text{Strong}}(3) \otimes \text{SU}_{\text{weak}}(2) \otimes \text{U}_Y(1)$ gauge theory without Higgs mechanism. Instead, there exist massive neutrinos with both left-hand and left-hand degrees of freedom.

	Quantum field theory of Standard model	Knot-crystal theory
Elementary particles	Electron, u-quark, d-quark, neutrino	Composite knots with 3, 2, 1, 0 internal-windings
Electromagnetic interaction	Gauge field of U(1) gauge symmetry	Position-fluctuations of internal-windings of a composite knot
Strong interaction	Gauge fields of SU(3) gauge symmetry	Order-fluctuations of internal-windings of a composite knot
Weak interaction	Gauge fields of SU(2) gauge symmetry	Phase and density fluctuations of internal-windings in big-ring
Parity breaking	No right-hand neutrino and chirality property of weak interaction	Mismatch between A-brane and B-brane
Mass	Higgs condensation	Finite rotating velocity of two branes

FIG. 22: Comparison between quantum field theory of Standard model and knot-crystal physics

G. Summary

In the end of this section, we give a summary on the knot-crystal theory. The knot-crystal theory provides a way to unify all gauge fields and elementary fermionic excitations – the composite knots can be regarded as bound states of a bare knot and n -internal-winding. The internal-windings can be regarded as interaction mediums. By exchanging quantum fluctuations of internal-windings, there exist gauge interaction between composite knots. In Fig.22, we compare quantum field theory of the Standard model and knot-crystal theory.

Let us summarize the key points of knot-crystal theory:

1. **Quantum fields as knot crystal:** Knot-crystal becomes the foundation of quantum fields, i.e.,

$$\text{Knot-crystals} \iff \text{Quantum fields}.$$

For a 1-level double-helix knot-crystal, the effective theory is Dirac model and the collective motions of knots are described by Dirac equation; for a 2-level double-helix knot-crystal, the effective theory is an interacting quantum field theory, in which Dirac fermions couples gauge fields; for the standard knot-crystal, the low energy effective model is the Standard model in particle physics – an $\text{SU}_{\text{Strong}}(3) \otimes \text{SU}_{\text{Weak}}(2) \otimes \text{U}_Y(1)$ gauge theory with Higgs mechanism due to spontaneous symmetry breaking.

2. **Gauge fields as quantum fluctuations of composite knot-crystal:** After mapping 2-level double-helix knot-crystal to composite knot-crystal, the composite knots become elementary particles in particle physics, including neutrino, electron and quarks. Gauge symmetries for composite knots are the redundancies from internal spatial translation symmetry for quantum states of composite knot-crystal. The collective quantum fluctuations of the composite knot-crystal become gauge fields – position-fluctuations of internal-windings are Abelian gauge field and order-fluctuations of internal-windings are non-Abelian gauge field. The following equation shows the gauge fields and fermionic particles

$$\begin{aligned}
A (\Delta_{12})_{A/B} &= n + \frac{1}{2} \text{ 2-level knot-crystal} \\
&= A (\Delta_{12})_{A/B} = \frac{1}{2} \text{ 2-level knot-crystal (fermionic fields)} \\
&+ a (\Delta_{12})_{A/B} = n \text{ 2-level knot-crystal (gauge fields)}.
\end{aligned} \tag{487}$$

3. **Electro-weak interaction and its chirality:** The electro-weak $SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge symmetry comes from the redundancies of 2-level composite linear knots in big-rings. The $SU_{\text{weak}}(2)$ gauge fields are the density-fluctuations of internal-windings on a big-ring and the $U_Y(1)$ gauge fields are the position-fluctuations of composite knots and internal-windings on a big-ring i.e.,

$$\begin{aligned} SU_{\text{weak}}(2) \text{ gauge field} &\implies \text{density-fluctuations of} \\ &\quad \text{internal-windings on a big-ring,} \\ U_Y(1) \text{ gauge field} &\implies \text{position-fluctuations of composite} \\ &\quad \text{knots and internal-windings on a big-ring.} \end{aligned}$$

The chirality property of weak $SU_{\text{weak}}(2) \otimes U_Y(1)$ interaction comes from intrinsic mismatch between two branes $(\Delta_{23})_A - (\Delta_{23})_B = \delta \neq 0$.

4. **Higgs field and Higgs mechanism:** The fluctuating rotating velocity of a standard knot-crystal $\omega_0 \rightarrow \omega(\vec{X}, t)$ plays the role of Higgs field $\Phi(\vec{X}, t)$ in SM. Because the rotating field $\omega(\vec{X}, t)$ is an $SU_{\text{weak}}(2)$ complex doublet and has supercharge $\mathcal{Y} = 1$, a finite rotating velocity of knot-crystal corresponds to Higgs condensation that gives masses to three vector gauge bosons and knots (electrons and quarks).
5. **Standard model and standard knot-crystal:** For a special knot-crystal with hierarchy series of universe $\left(\begin{smallmatrix} \{3.5, N\} \\ \{3.5, N + \delta\} \end{smallmatrix} \right)$ – standard knot-crystal, the low energy effective theory is just the Standard model – an $SU_{\text{Strong}}(3) \otimes SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge theory with Higgs mechanism due to spontaneous symmetry breaking.

IV. GRAVITY THEORY IN KNOT PHYSICS – CHIRAL GRAVITY THEORY

Gravity is a natural phenomenon by which all objects attract one another including galaxies, stars, human-being and even elementary particles. Hundreds of years ago, Newton discovered the inverse-square law of universal gravitation,

$$F = \frac{GMm}{r^2} \quad (488)$$

where G is the Newton constant, r is the distance, and M and m are the possess masses for two objects. One hundred years ago, the establishment of general relativity by Einstein is a milestone to learn the underline physics of gravity that provides a unified description of gravity as a geometric property of spacetime. From Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad (489)$$

the gravitational force is really an effect of curved spacetime. Here $R_{\mu\nu}$ is the 2nd rank Ricci tensor, R is the curvature scalar, $g_{\mu\nu}$ is the metric tensor, and $T_{\mu\nu}$ is the energy-momentum tensor of matter. c is speed of light.

In the framework of quantum field theory, it is believed that the gravitational interaction comes from exchanging virtual gravitons - spin-2 bosonic particles. The primary approach to quantum gravity leads to unsolvable divergences. Based on different principles, there are different approaches to quantum gravity, including topological BF (gauge) theory for the Lorentz group[32, 33], superstring theory[2], quantum loop theory[4] and twistor theory[34]. However, gravity is still very mysterious and far from being well understood. Black hole is a big monster to be recognized, of which there are a lot of unsolved mysteries including singularity problem and quantum information problem. It is known that dark energy (or the cosmological constant Λ) makes up 68% of universal density. The origin of dark energy is a challenge for quantum gravity theory.

The study of the knot-crystal and its dynamics suggest a new approach to grand unified theory that unify all three non-gravitational interactions: weak interaction, strong interaction, and electromagnetic interaction. The composite knot can be regarded as a bound state of a bare knot and internal-windings. By exchanging quantum fluctuations of internal-windings, there exists gauge interaction between composite knots. In particular, the quantum field theory for the grand unified theories has no divergence due to a natural cutoff. As a result, the knot physics becomes a new candidate to theory-of-everything (TOE). To reach TOE, we generalize knot-crystal theory to gravitational interaction and derive a theory of quantum gravity.

In this part, we try to answer question 5 and question 6 and derive a new theory of gravity (we call it chiral-gravity theory). We point out that gravity is an emergent phenomenon in the framework of knot-physics. Based on three dimensional (3D) double-helix-knot-crystal with chiral-density-wave order (*chiral-spacetime-crystal*) in five dimensional

(5D) space, the gravitational interaction comes from (topological) chiral-orbital coupling effect by unbalancing the two-branes in chiral-spacetime-crystal. On the one hand, the unbalance of two branes in a chiral-spacetime-crystal can be denoted by curved spacetime; on the other hand, the knots unbalance the two branes in chiral-spacetime-crystal that indicates matter may curve spacetime. Now, quantum fields of the Standard model and curved spacetime are unified into a simple phenomenon – composite knot-crystal (standard knot-crystal).

This part is organized as below. In Sec. A, we review knot-crystal theory. In Sec. B, we discuss the unbalancedly-rotating knot-crystal and show coordinate transformation in quantum physics. In Sec. C, we study chiral angle – a new degree of freedom of a 3D 1-level double-helix knot-crystal that corresponds to a global mismatch of two branes. To characterize the spacial distribution of inter-brane phase, we define chiral order and chiral transformation. In Sec. D, we introduce a key concept in chiral-gravity physics – chiral-spacetime-crystal that is an unbalancedly-rotating knot-crystal with chiral-density-wave (chiral-DW) order. In Sec. E, by setting the coordinates of the effective "spacetime" to the chiral angles, $(\vec{\Theta}, \Theta_t)$, we found that the fluctuations of chiral-density-wave order lead to deformed chiral-spacetime-crystal that are described by an emergent curved spacetime. An important property of a deformed chiral-spacetime-crystal is diffeomorphism invariance. In this section, we also show the relationship between the gauge-field-representation for non-uniform chiral-density-wave order and the geometry-representation for curved spacetime. In Sec. F, the gravity in knot physics is explored to be a topological chiral-orbital coupling effect. In this section, we develop a complete theory for gravity – chiral-gravity theory. In Sec. G, we apply the chiral-gravity theory on the standard knot-crystal and point out an establishment of a great unification theory of all fundamental interactions. Finally, the conclusions are drawn in Sec. H.

A. Review on knot-crystal

Knot-crystal is a periodic entanglement-pattern between two d-branes. The key property of knot-crystals is generalized translation symmetry. To classify a knot-crystal, we had introduced four indices: dimension d , brane-number \mathcal{N} , level \mathcal{M} and chirality L/R. Different 3D knot-crystals are characterized by different hierarchy recurrence relationships.

An typical knot-crystal is an $\mathcal{N} = 2$ 1-level double-helix knot-crystal, there are four possible chiralities, i.e.,

$$\mathbf{Z}_{LL}(\vec{x}, t) = \begin{pmatrix} r_A e^{i\vec{k}_0 \cdot \vec{x} + i\omega_A t + i\phi_{0,A}} \\ r_B e^{i\vec{k}_0 \cdot \vec{x} + i\omega_B t + i\phi_{0,B}} \end{pmatrix}, \quad (490)$$

$$\mathbf{Z}_{LR}(\vec{x}, t) = \begin{pmatrix} r_A e^{i\vec{k}_0 \cdot \vec{x} + i\omega_A t + i\phi_{0,A}} \\ r_B e^{-i\vec{k}_0 \cdot \vec{x} + i\omega_B t + i\phi_{0,B}} \end{pmatrix}, \quad (491)$$

$$\mathbf{Z}_{RL}(\vec{x}, t) = \begin{pmatrix} r_A e^{-i\vec{k}_0 \cdot \vec{x} + i\omega_A t + i\phi_{0,A}} \\ r_B e^{i\vec{k}_0 \cdot \vec{x} + i\omega_B t + i\phi_{0,B}} \end{pmatrix}, \quad (492)$$

$$\mathbf{Z}_{RR}(\vec{x}, t) = \begin{pmatrix} r_A e^{-i\vec{k}_0 \cdot \vec{x} + i\omega_A t + i\phi_{0,A}} \\ r_B e^{-i\vec{k}_0 \cdot \vec{x} + i\omega_B t + i\phi_{0,B}} \end{pmatrix}, \quad (493)$$

where $\phi_{0,A/B}$ is a constant angle for A/B-brane. $\vec{k}_0 = \frac{\pi}{a}\vec{e}$, r_0 is constant, a is a fixed length that denotes the half pitch of windings and \vec{e} is a given direction in 3D space. In general, r_A and r_B are different $r_A \neq r_B$ but very close, $r_A \simeq r_B$.

For a special type of rotating symmetric double-helix knot-crystal, we introduced the concept of balancedly-rotating knot-crystal, of which all knots are static at given positions with periodically time-changing of brane indices. There are two types of balancedly-rotating knot-crystals: symmetric double-helix knot-crystal with $\omega_A = -\omega_B = \omega$ and anti-symmetric double-helix knot-crystal with $\omega_A = \omega_B = \omega$. The constant rotating velocity $|\omega_A| = |\omega_B| = \omega_0$ of balancedly-rotating knot-crystals plays the role of mass of Dirac fields ($mc^2 = \hbar\omega_0$).

The low energy effective model of 3D 1-level balancedly-rotating knot-crystals is Dirac model, of which the effective Hamiltonian is given by

$$\mathcal{H}_{3D} = \int (\psi^\dagger \hat{H}_{3D} \psi) d^3x \quad (494)$$

where

$$\begin{aligned}\hat{H}_{3D} &= c(\tau_y \otimes \vec{\sigma}) \cdot \vec{p} + m(\tau_x \otimes \mathbf{1}) \\ &= c\vec{\Gamma} \cdot \vec{p} + m\Gamma^5\end{aligned}\quad (495)$$

and $\psi^\dagger = (\psi_{1\uparrow}^\dagger, \psi_{2\uparrow}^\dagger, \psi_{1\downarrow}^\dagger, \psi_{2\downarrow}^\dagger)$ are the generation operators of four-component knots. $\vec{p} = -i\vec{\nabla}$ is 3D momentum operator and $\vec{\Gamma} = \tau_y \otimes \vec{\sigma} = (\Gamma^1, \Gamma^2, \Gamma^3)$ is 4-by-4 Gamma matrix vector. Γ^a satisfy the Clifford algebra $\{\Gamma^a, \Gamma^b\} = 2\delta_{ab}\mathbb{I}$ with \mathbb{I} the identity 4-by-4 matrix ($a = 1, 2, 3, 4, 5$). c is light velocity. 1 and 2 denote the sublattice indices.

Thus, the effective Lagrangian of the 3D 1-level balancedly-rotating knot-crystals is written as

$$\mathcal{L}_{3D} = \bar{\psi}(i\gamma^\mu \hat{\partial}_\mu - m)\psi \quad (496)$$

where $\bar{\psi} = \psi^\dagger \gamma^0$. γ^μ are the reduced Gamma matrices, $\gamma^1 = \gamma^0 \Gamma^1$, $\gamma^2 = \gamma^0 \Gamma^2$, $\gamma^3 = \gamma^0 \Gamma^3$, and $\gamma^0 = \Gamma^5 = \tau_x \otimes \mathbf{1}$, $\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$. This is the Lagrangian for massive Dirac fermions with emergent SO(3,1) Lorentz-invariance. The SO(3,1) Lorentz transformations S_{Lor} is defined by

$$S_{\text{Lor}} \gamma^\mu S_{\text{Lor}}^{-1} = (\gamma^\mu)', \quad (\mu = 0, 1, 2, 3) \quad (497)$$

and

$$S_{\text{Lor}} \gamma^5 S_{\text{Lor}}^{-1} = \gamma^5. \quad (498)$$

B. Unbalancedly-rotating knot-crystals and chiral-time-crystal

In this section, based on a 3D double-helix knot-crystal, we introduce the concept of "unbalancedly-rotating knot-crystal". An unbalancedly-rotating knot-crystal is a 3D double-helix knot-crystal by unbalancing the rotating velocities of two branes, i.e., $\omega_A \neq -\omega_B$ for symmetric 3D double-helix knot-crystal and $\omega_A \neq \omega_B$ for anti-symmetric 3D double-helix knot-crystal. In physics, on an unbalancedly-rotating knot-crystal, all knots move with same velocity. This phenomenon corresponds to coordinate transformation in quantum physics. All quantum states on an unbalancedly-rotating knot-crystal correspond to those on a balancedly-rotating knot-crystal under a Lorentz boost S_{Lor} .

We take a 1-level double-helix knot-crystal $\mathbf{Z}_{\text{LL}}(\vec{x}, t)$ as an example to show unbalancedly-rotating knot-crystal. An $\mathcal{N} = 2$ 1-level double-helix knot-crystal is described by

$$\mathbf{Z}_{\text{LL}}(\vec{x}, t) = \begin{pmatrix} z_A(\vec{x}, t) \\ z_B(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} r_A e^{i\phi_A(\vec{x}, t)} \\ r_B e^{i\phi_B(\vec{x}, t)} \end{pmatrix}, \quad (499)$$

where

$$\phi_A(\vec{x}, t) = \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \omega_A t + \phi_{0,A} \quad (500)$$

and

$$\phi_B(\vec{x}, t) = \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \omega_B t + \phi_{0,B}. \quad (501)$$

After out-brane-projection, $\hat{P}_\theta[z_A(x)] = \hat{P}_\theta[z_B(x)]$, we have the knot-equation $\hat{P}_\theta[z_A(x)] = \hat{P}_\theta[z_B(x)]$ to be

$$\begin{aligned}\vec{x}' &= a \cdot n - \frac{a}{2\pi}(\omega_A + \omega_B)t \\ &\quad - \frac{a}{2\pi}(\phi_{0,A} + \phi_{0,B}).\end{aligned} \quad (502)$$

For the symmetric double-helix knot-crystal with $\omega_A = -\omega_B = \omega$, the knot-equation can be simplified into

$$\vec{x}' = a \cdot n - \frac{a}{2\pi}(\phi_{0,A} + \phi_{0,B}) \quad (503)$$

where $x' = \vec{x} \cdot \vec{e}$ is the coordination on the axis along a given direction \vec{e} and n is an integer number.

1. Chiral-time-crystal

We then consider an unbalancedly-rotating knot-crystal as a combination of a balancedly-rotating (Dirac) knot-crystal $\mathbf{Z}_{\text{Dirac}}(\vec{x}, t)$ and a chiral-time-crystal $\mathbf{Z}_{\text{chiral-time}}$, i.e.,

$$\begin{aligned}\mathbf{Z}_{\text{LL}}(\vec{x}, t) &= \begin{pmatrix} z_{\text{A}}(\vec{x}, t) \\ z_{\text{B}}(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} r_{\text{A}} e^{i\phi_{\text{A}}(\vec{x}, t)} \\ r_{\text{B}} e^{i\phi_{\text{B}}(\vec{x}, t)} \end{pmatrix} \\ &= \mathbf{Z}_{\text{Dirac}}(\vec{x}, t) \otimes \mathbf{Z}_{\text{chiral-time}}\end{aligned}\quad (504)$$

where the balancedly-rotating knot-crystal $\mathbf{Z}_{\text{Dirac}}(\vec{x}, t)$ is defined by

$$\mathbf{Z}_{\text{Dirac}}(\vec{x}, t) = \begin{pmatrix} r_{\text{A}} e^{i[\frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \omega t + \phi_{0, \text{A}}]} \\ r_{\text{B}} e^{i[\frac{\pi}{a}(\vec{e} \cdot \vec{x}) - \omega t + \phi_{0, \text{B}}]} \end{pmatrix}, \quad (505)$$

and the chiral-time-crystal $\mathbf{Z}_{\text{chiral-time}}(\vec{x}, t)$ is defined by

$$\mathbf{Z}_{\text{chiral-time}}(\vec{x}, t) = \begin{pmatrix} r_{\text{A}} \\ r_{\text{B}} e^{i\Delta\omega \cdot t} \end{pmatrix}. \quad (506)$$

For an $\mathcal{N} = 2$ 1-level double-helix knot-crystal $\mathbf{Z}_{\text{LL}}(\vec{x}, t)$, we call $\omega = \frac{\omega_{\text{A}} - \omega_{\text{B}}}{2}$ to be global rotating velocity and $\Delta\omega = \omega_{\text{B}} + \omega_{\text{A}}$ to global velocity.

The chiral-time-crystal $\mathbf{Z}_{\text{chiral-time}}(\vec{x}, t)$ denotes a knot-crystal of brane-B with constant rotating velocity. Because the phase angle difference $\Theta(\vec{x}, t)$ between two branes is proportion to time, $\Theta(\vec{x}, t) = \phi_{\text{A}}(\vec{x}, t) - \phi_{\text{B}}(\vec{x}, t) = \Delta\omega \cdot t$, we call $\mathbf{Z}_{\text{chiral-time}}(\vec{x}, t)$ to be *chiral-time-crystal*. In mathematic, the definition of a chiral-time-crystal is based on time-translation operation $\mathcal{T}(\Delta t)$, i.e.,

$$\begin{aligned}\mathcal{T}(\Delta t)z_{\text{A}}(\vec{x}, t) &= e^{i\Delta\phi_{\text{A}}} z_{\text{A}}(\vec{x}, t), \\ \mathcal{T}(\Delta t)z_{\text{B}}(\vec{x}, t) &= e^{i\Delta\phi_{\text{B}}} z_{\text{B}}(\vec{x}, t).\end{aligned}\quad (507)$$

For a chiral-time-crystal, the chiral angle is obtained as

$$\begin{aligned}\Theta(\vec{x}, t) &= \phi_{\text{A}}(\vec{x}, t) - \phi_{\text{B}}(\vec{x}, t) \\ &= \Delta\phi_{\text{A}} - \Delta\phi_{\text{B}} = \alpha\Delta t\end{aligned}\quad (508)$$

where α is a constant value, $\alpha = \Delta\omega$.

Due to the superposition principle, we may use the generation operators of knot-crystals to create an unbalancedly-rotating knot-crystal. We define the knot-crystal-operations of composite unbalancedly-rotating knot-crystal $\mathbf{Z}_{\text{LL}}(\vec{x}, t)$, balancedly-rotating knot-crystal $\mathbf{Z}_{\text{Dirac}}(\vec{x}, t)$ and chiral-time-crystal $\mathbf{Z}_{\text{chiral-time}}(\vec{x}, t)$ by $\hat{U}(\mathbf{Z}_{\text{LL}})$, $\hat{U}(\mathbf{Z}_{\text{Dirac}})$, and $\hat{U}(\mathbf{Z}_{\text{chiral-time}})$, respectively. We could derive the knot-functions by corresponding knot-crystal-operations, i.e.,

$$\begin{aligned}\mathbf{Z}_{\text{LL}}(\vec{x}, t) &= \hat{U}(\mathbf{Z}_{\text{LL}}) \begin{pmatrix} r_{\text{A}} \\ r_{\text{B}} \end{pmatrix}, \\ \mathbf{Z}_{\text{Dirac}}(\vec{x}, t) &= \hat{U}(\mathbf{Z}_{\text{Dirac}}) \begin{pmatrix} r_{\text{A}} \\ r_{\text{B}} \end{pmatrix}, \\ \mathbf{Z}_{\text{chiral-time}}(\vec{x}, t) &= \hat{U}(\mathbf{Z}_{\text{chiral-time}}) \begin{pmatrix} r_{\text{A}} \\ r_{\text{B}} \end{pmatrix}.\end{aligned}\quad (509)$$

In particular, the knot-crystal-operation of chiral-time-crystal $\mathbf{Z}_{\text{chiral-time}}(\vec{x}, t)$ is obtained as

$$\hat{U}(\mathbf{Z}_{\text{chiral-time}}) = \begin{pmatrix} 1 \\ \exp[i\phi_{\text{B}}(t) \cdot \hat{K}] \end{pmatrix} \quad (510)$$

where $\hat{K} = -i\frac{d}{d\phi_{\text{B}}}$ and $\phi_{\text{B}}(t) = \Delta\omega \cdot t$.

Thus, the combination of the unbalancedly-rotating knot-crystal $\mathbf{Z}_{\text{LL}}(\vec{x}, t)$ by the balancedly-rotating knot-crystal $\mathbf{Z}_{\text{Dirac}}(\vec{x}, t)$ and the chiral-time-crystal $\mathbf{Z}_{\text{chiral-time}}(\vec{x}, t)$ is represented by

$$\hat{U}(\mathbf{Z}_{\text{LL}}) = \hat{U}(\mathbf{Z}_{\text{chiral-time}}) \cdot \hat{U}(\mathbf{Z}_{\text{Dirac}}). \quad (511)$$

2. Inertial system for the case of uniform $\Delta\omega$

For the unbalancedly-rotating knot-crystal with $\omega_A + \omega_B = \Delta\omega$, the knot-solution is given by

$$\begin{aligned}\vec{x}'(t) &= a \cdot n - \frac{a}{2\pi}(\omega_A + \omega_B)t - \frac{a}{2\pi}(\phi_{0,A} + \phi_{0,B}) \\ &= a \cdot n - \frac{a}{2\pi}\Delta\omega \cdot t - \frac{a}{2\pi}(\phi_{0,A} + \phi_{0,B}).\end{aligned}\quad (512)$$

We have

$$\vec{v} = \frac{d\vec{x}'(t)}{dt} = -\frac{a}{2\pi}\Delta\omega \cdot \vec{e}.$$

For a knot-crystal with a global velocity $\vec{v} = -\frac{a}{2\pi}\Delta\omega \cdot \vec{e}$, due to SO(3,1) Lorentz-invariance, we can do Lorentz boosting by unbalancing the rotating velocities of two branes,

$$t \rightarrow t' = \frac{t - \vec{x} \cdot \vec{v}/c^2}{\sqrt{1 - \vec{v}^2}}, \quad (513)$$

$$\vec{x} \rightarrow \vec{x}' = \frac{\vec{x} - \vec{v} \cdot t}{\sqrt{1 - \vec{v}^2}}. \quad (514)$$

As a result, we may get the *inertial reference-frame* for all quantum states under Lorentz boost by uniformly unbalancing the rotating velocities of two branes, i.e.,

$$\psi(\vec{x}, t) \rightarrow \psi'(\vec{x}', t') = S_{\text{Lor}} \cdot \psi(\vec{x}', t'). \quad (515)$$

For a particle-like excitation on the balancedly-rotating knot-crystal $\mathbf{Z}_{\text{Dirac}}(\vec{x}, t)$, the wave-function $\psi(t)$ is $\frac{1}{\sqrt{V}}e^{-i\omega t}$. Under coordinate transformation with small velocity $|\vec{v}|$, this wave-function $\psi(t)$ is transformed into

$$\begin{aligned}\psi(t) &= \frac{1}{\sqrt{V}}e^{-i\omega t} \rightarrow \psi' = \frac{1}{\sqrt{V}}e^{-i\omega t'} \\ &\simeq \frac{1}{\sqrt{V}}e^{-i\omega t} \exp(-i(Et - \vec{p} \cdot \vec{x}))\end{aligned}\quad (516)$$

where $E \simeq \frac{\vec{p}^2}{2m}$, $\vec{p} \simeq \omega \frac{\vec{v}}{c^2}$ and $mc^2 = \hbar\omega$.

As a result, we derive the waving patterns of a linear knot by doing waving brane-twisting, that are described by the plane-wave wave-function $\frac{1}{\sqrt{V}}e^{-i\omega t} \exp(-i(Et - \vec{p} \cdot \vec{x}))$. The spacial-dependence of wave-function $\exp(-i(Et - \vec{p} \cdot \vec{x}))$ comes from the contribution of $\hat{U}(\mathbf{Z}_{\text{chiral-time}})$ that generates a (uniform) chiral-time-crystal and thus corresponds to Lorentz boosting S_{Lor} ,

$$\hat{U}(\mathbf{Z}_{\text{chiral-time}}) \implies \text{Lorentz boosting } S_{\text{Lor}}. \quad (517)$$

So, we have

$$\begin{aligned}\text{Chiral-time-crystal} \\ \implies \text{Inertial system.}\end{aligned}\quad (518)$$

3. Noninertial system for the case of non-uniform $\Delta\omega(\vec{x}, t)$

Noninertial system with local Lorentz boost can be obtained by non-uniformly-unbalancing the rotating velocities of two branes, i.e., $\Delta\omega(\vec{x}, t)$. We get a non-uniform velocity of knots

$$\vec{v}(\vec{x}, t) = -\frac{a}{2\pi}\Delta\omega(\vec{x}, t) \cdot \vec{e}(\vec{x}, t). \quad (519)$$

Due to local SO(3,1) Lorentz-invariance, we can locally do Lorentz boosting by unbalancing the two branes,

$$\begin{aligned}t \rightarrow t'(\vec{x}, t) &= \frac{t - \vec{x} \cdot \vec{v}(\vec{x}, t)/c^2}{\sqrt{1 - (\vec{v}(\vec{x}, t))^2}}, \\ \vec{x} \rightarrow \vec{x}'(\vec{x}, t) &= \frac{\vec{x} - \vec{v}(\vec{x}, t) \cdot t}{\sqrt{1 - (\vec{v}(\vec{x}, t))^2}}.\end{aligned}\quad (520)$$

Thus, a chiral knot-crystal described by $\mathbf{Z}_{\text{chiral-time}}(t)$ (or $\hat{U}(\mathbf{Z}_{\text{chiral-time}})$) turns into a *non-uniform-chiral-time-crystal* described by $\mathbf{Z}_{\text{non-uniform-chiral-time}}(\vec{x}', t')$ (or $\hat{U}(\mathbf{Z}_{\text{non-uniform-chiral-time}})$) as

$$\begin{aligned}\mathbf{Z}_{\text{chiral-time}}(\vec{x}, t) &= \begin{pmatrix} r_A \\ r_B e^{i\Delta\omega \cdot t} \end{pmatrix} \\ &\rightarrow \mathbf{Z}_{\text{non-uniform-chiral-time}}(\vec{x}', t') \\ &= \hat{U}(\mathbf{Z}_{\text{non-uniform-chiral-time}}) \begin{pmatrix} r_A \\ r_B \end{pmatrix} \\ &= \begin{pmatrix} r_A \\ r_B e^{i\Delta\omega(\vec{x}, t) \cdot t} \end{pmatrix}.\end{aligned}\quad (521)$$

We may do non-uniform Lorentz boosting $S_{\text{Lor}}(\vec{x}, t)$ by non-uniformly unbalancing the rotating velocities of two branes, i.e.,

$$\begin{aligned}\psi(\vec{x}, t) &\rightarrow \psi'(\vec{x}'(\vec{x}, t), t'(\vec{x}, t)) \\ &= S_{\text{Lor}}(\vec{x}, t)\psi(\vec{x}, t).\end{aligned}\quad (522)$$

The spacial-dependence of wave-functions of all quantum states changes following the non-uniform knot-crystal-operation $\hat{U}(\mathbf{Z}_{\text{non-uniform-chiral-time}})$ via non-uniform Lorentz boosting $S_{\text{Lor}}(\vec{x}, t)$.

Due to the existence of non-uniform Lorentz invariance, noninertial system emerges. So, we have

$$\begin{aligned}\text{Non-uniform-chiral-time-crystal} \\ \implies \text{noninertial system}.\end{aligned}\quad (523)$$

C. Chiral order for knot-crystal

In addition to global velocity, there exists another important dynamic degree of freedom of a 3D 1-level double-helix knot-crystal – spacial distribution of inter-brane phase $\Theta = \phi_{0,A} - \phi_{0,B}$ that corresponds to a global mismatch between two branes. To characterize the spacial distribution of inter-brane phase, we introduce three concepts of 3D 1-level double-helix knot-crystal: *chiral angle*, *chiral order* and *chiral transformation*.

Firstly, we define the inter-brane phase $\Theta = \phi_{0,A} - \phi_{0,B}$ to be *chiral angle*. We call a 3D 1-level double-helix knot-crystal with uniform spacial distribution of chiral angle ($\Theta = \text{constant}$) to be *uniform chiral order*. In Fig.23, we show two different uniform chiral orders. Fig.23.(a) and Fig.23.(d) show uniform chiral order for the knot-crystal with $\Theta \equiv \pi$ and Fig.23.(b) and Fig.23.(e) show uniform chiral order for the knot-crystal with $\Theta \neq \pi$. The 3D 1-level double-helix knot-crystal with non-uniform spacial distribution of chiral angle is called *non-uniform chiral order* $\Theta(\vec{x}, t)$ (See the illustration in Fig.23.(c) and Fig.23.(f)).

1. Chiral transformation

Under a uniform chiral transformation, the knot-function of knot-crystal becomes

$$\begin{aligned}\mathbf{Z}_{\text{Dirac}}(\vec{x}, t) &\rightarrow \mathbf{Z}'_{\text{Dirac}}(\vec{x}, t) = \begin{pmatrix} z'_A(\vec{x}, t) \\ z'_B(\vec{x}, t) \end{pmatrix} \\ &= \begin{pmatrix} z_A(\vec{x}, t)e^{i\Theta/2} \\ z_B(\vec{x}, t)e^{-i\Theta/2} \end{pmatrix} \\ &= \begin{pmatrix} r_A e^{i\phi_A(\vec{x}, t) + i\Theta/2} \\ r_B e^{i\phi_B(\vec{x}, t) - i\Theta/2} \end{pmatrix}\end{aligned}\quad (524)$$

where

$$\phi_A(x, t) = \pm \frac{\pi}{a}x + \omega_A t + \phi_{0,A} \quad (525)$$

and

$$\phi_B(x, t) = \pm \frac{\pi}{a}x + \omega_B t + \phi_{0,B}. \quad (526)$$

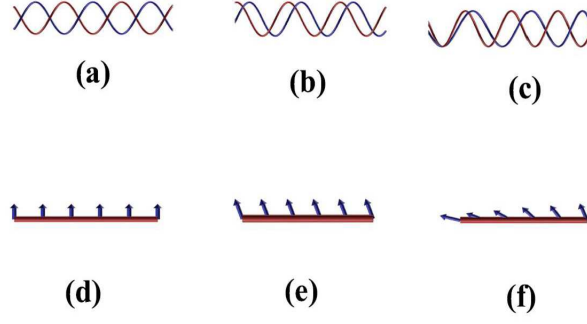


FIG. 23: (a) An illustration of a 1D double-helix knot-crystal with uniform chiral order ($\Theta = \pi$); (b) An illustration of a 1D double-helix knot-crystal with uniform chiral order ($\Theta \neq \pi$); (c) An illustration of a 1D double-helix knot-crystal with non-uniform chiral order; (d) An illustration of uniform chiral order ($\Theta = \pi$); (e) An illustration of uniform chiral order ($\Theta \neq \pi$); (f) An illustration of non-uniform chiral order. The blue arrows denote the directions of chiral orders.

To characterize different chiral orders, we define *chiral transformation* for quantum states to be

$$\hat{U}_{\text{chiral}}(\Theta) = \exp(i(\Theta \cdot \tau_z \otimes \vec{\sigma}/2))$$

where $\vec{\sigma}$ is spin operator along \vec{e} direction. So, by setting $\vec{\sigma} = \sigma^z$, we have

$$\begin{aligned} \hat{U}_{\text{chiral}}(\Theta) \begin{pmatrix} |A, \uparrow\rangle \\ |A, \downarrow\rangle \\ |B, \uparrow\rangle \\ |B, \downarrow\rangle \end{pmatrix} &= \begin{pmatrix} e^{i\Theta/2} & 0 & 0 & 0 \\ 0 & e^{-i\Theta/2} & 0 & 0 \\ 0 & 0 & e^{-i\Theta/2} & 0 \\ 0 & 0 & 0 & e^{i\Theta/2} \end{pmatrix} \begin{pmatrix} |A, \uparrow\rangle \\ |A, \downarrow\rangle \\ |B, \uparrow\rangle \\ |B, \downarrow\rangle \end{pmatrix} \\ &= \begin{pmatrix} e^{i\Theta/2} |A, \uparrow\rangle \\ e^{-i\Theta/2} |A, \downarrow\rangle \\ e^{-i\Theta/2} |B, \uparrow\rangle \\ e^{i\Theta/2} |B, \downarrow\rangle \end{pmatrix}. \end{aligned} \quad (527)$$

For a 3D double-helix knot-crystal, under a spacial chiral transformation along a given direction \vec{e} , we have

$$\begin{aligned} \hat{U}_{\text{chiral}}(\Theta) |A\rangle &= |A'\rangle = \hat{T}(\pm\Delta x) |A\rangle, \\ \hat{U}_{\text{chiral}}(\Theta) |B\rangle &= |B'\rangle = \hat{T}(\pm\Delta x) |B\rangle, \end{aligned} \quad (528)$$

where $\hat{T}(\Delta x)$ is the translation operator $\hat{T}(\Delta x) |i, A/B\rangle = |i + \Delta x, A/B\rangle$ that is

$$\hat{T}(\Delta x) = e^{i\Delta x(\hat{k} \cdot \vec{\sigma})} \quad (529)$$

and

$$\Delta x = \Theta \cdot \frac{a}{2\pi}. \quad (530)$$

Due to finite phase mismatch, the position of knot-solutions changes. So, a spacial chiral transformation leads to a special spacial shift for the knot-crystal. From similar argument, the effect of unbalancedly-rotating knot-crystal can be regarded a temporal chiral transformation.

2. Chiral order

For a 1-level double-helix knot-crystal, under a spacial chiral transformation $U_{\text{chiral}}(t, \vec{x})$, we have

$$\begin{aligned}\Gamma^5 &= \tau_x \otimes \mathbf{1} \\ &\rightarrow U_{\text{chiral}}(t, \vec{x}) \Gamma^5 (U_{\text{chiral}}(t, \vec{x}))^{-1} = (\Gamma^5(t, \vec{x}))'.\end{aligned}\tag{531}$$

Different chiral orders are defined by different $U_{\text{chiral}}(t, \vec{x})$ on Γ^a , i.e.,

$$(\Gamma^a(t, \vec{x}))' = U_{\text{chiral}}(t, \vec{x}) \Gamma^a (U_{\text{chiral}}(t, \vec{x}))^{-1}.\tag{532}$$

To characterize different types of chiral orders $(\Gamma^a(t, \vec{x}))'$, we introduce a classical unit vector-field $n^a(t, \vec{x})$ by

$$(\Gamma^5(t, \vec{x}))' = \sum_a \Gamma^a n^a(t, \vec{x})\tag{533}$$

where $\mathbf{n} = (n^1, n^2, n^3, \phi_0^5) = (\vec{n}, \phi_0^5)$ is an SO(4) rotor that denotes the direction of chiral order.

In addition, we point out that for a 3D 1-level double-helix knot-crystal with non-uniform chiral order, the spacial shifts for brane A and brane B become

$$\Delta x_A(\vec{x}, t) = \Theta(\vec{x}, t) \cdot \frac{a}{2\pi}\tag{534}$$

and

$$\Delta x_B(\vec{x}, t) = -\Theta(\vec{x}, t) \cdot \frac{a}{2\pi}.\tag{535}$$

As a result, the length of brane A is different from the length of brane B, i.e.,

$$\begin{aligned}\Delta L &= L_A - L_B \\ &= \int_{-\infty}^{\infty} \Delta x_A(\vec{x}, t) - \int_{-\infty}^{\infty} \Delta x_B(\vec{x}, t) \\ &= \frac{a}{\pi} \int d\Theta \\ &= \frac{a}{\pi} \left(\int_{-\infty}^{\infty} d\phi_{0,A} - \int_{-\infty}^{\infty} d\phi_{0,B} \right) \neq 0,\end{aligned}\tag{536}$$

where L_A is the length of brane A and L_B is the length of brane B. Thus, a non-uniform chiral order leads to non-uniform changing of spacial-dependence of all quantum states, i.e.,

$$\begin{aligned}\text{Non-uniform chiral order} \\ \implies \text{nonlinear spacial-changing of } \Theta(\vec{x}).\end{aligned}\tag{537}$$

3. Low energy effective Lagrangian of 3D balancedly-rotating knot-crystal with non-uniform chiral order

In this part, we derive the low energy effective Lagrangian of 3D 1-level balancedly-rotating knot-crystal with non-uniform chiral order. For a 3D 1-level balancedly-rotating knot-crystal with non-uniform chiral order, there also exist two types of energy costs – energy cost from curving a brane and that from rotating the knot-crystal. The coupling energy J between two nearest-neighbor knots comes from curving the branes and the rotating angular velocity is assumed to be $\omega = |\omega_A| = |\omega_B|$.

We define the localized quantum states of 3D 1-level balancedly-rotating knot-crystal with non-uniform chiral order $\hat{U}_{\text{chiral}}(t, \vec{x})$ to be $|x, A\rangle'$ and $|x, B\rangle'$. The quantum localized quantum state of 3D 1-level balancedly-rotating knot-crystal with non-uniform chiral order $|x, A\rangle'$ and $|x, B\rangle'$ can be obtained by a local chiral transformation $U_{\text{chiral}}(t, \vec{x})$ on the quantum localized states of 3D 1-level balancedly-rotating knot-crystal with uniform chiral order $|x, A\rangle$ and $|x, B\rangle$ i.e.,

$$\begin{pmatrix} |x, A\rangle' \\ |x, B\rangle' \end{pmatrix} = \hat{U}_{\text{chiral}}(t, \vec{x}) \begin{pmatrix} |x, A\rangle \\ |x, B\rangle \end{pmatrix}.\tag{538}$$

Under the chiral transformation $\hat{U}_{\text{chiral}}(t, \vec{x})$, we write down the Hamiltonian of a 3D 1-level balancedly-rotating knot-crystal with non-uniform chiral order as

$$\begin{aligned}
\mathcal{H}_{3D} &\simeq c \int \psi'^{\dagger} [(\tau_y \otimes \vec{\sigma}) \cdot \hat{k}] \psi' d^3x \\
&+ m \int \psi'^{\dagger} \tau_x \psi' d^3x \\
&= c \int \psi^{\dagger} [(\hat{U}_{\text{chiral}}(t, \vec{x}) \tau_y \otimes \vec{\sigma} \cdot \hat{k} (U_{\text{chiral}}(t, \vec{x}))^{-1})] \psi d^3x \\
&+ m \int \psi^{\dagger} \hat{U}_{\text{chiral}}(t, \vec{x}) \tau_x (U_{\text{chiral}}(t, \vec{x}))^{-1} \psi d^3x
\end{aligned} \tag{539}$$

where $mc^2 = \hbar\omega$ plays role of the mass of knots and

$$\psi'(t, \vec{x}) = \begin{pmatrix} \psi'_{A\uparrow}(t, \vec{x}) \\ \psi'_{B\uparrow}(t, \vec{x}) \\ \psi'_{A\downarrow}(t, \vec{x}) \\ \psi'_{B\downarrow}(t, \vec{x}) \end{pmatrix} = \hat{U}_{\text{chiral}}(t, \vec{x}) \begin{pmatrix} \psi_{A\uparrow}(t, \vec{x}) \\ \psi_{B\uparrow}(t, \vec{x}) \\ \psi_{A\downarrow}(t, \vec{x}) \\ \psi_{B\downarrow}(t, \vec{x}) \end{pmatrix} \tag{540}$$

is a four-component fermion field of quantum state of 3D 1-level balancedly-rotating knot-crystal with non-uniform chiral order. A, B label chiral (brane) degrees of freedom and \uparrow, \downarrow label two spin degrees of freedom that denote the two possible winding directions along a given direction \vec{e} . In the following parts, we set the light velocity $c = aJ$ to be unit, i.e., $c = 1$.

For the case of uniform chiral order, due to $\hat{U}_{\text{chiral}}(t, \vec{x}) = \text{constant}$, we have

$$\mathcal{H}_{3D} = \int [\psi^{\dagger} (\vec{\Gamma} \cdot \vec{p} + m\Gamma^5) \psi] d^3x. \tag{541}$$

For the case of non-uniform chiral order, due to $\hat{U}_{\text{chiral}}(t, \vec{x}) \neq \text{constant}$, we have

$$\mathcal{H}_{3D} = \int [\psi^{\dagger} (\vec{\Gamma}'(t, \vec{x}) \cdot (\vec{p} - \vec{A}(t, \vec{x})) + m(\Gamma^5(t, \vec{x}))') \psi] d^3x \tag{542}$$

where

$$\begin{aligned}
\vec{\Gamma} &\rightarrow \vec{\Gamma}'(t, \vec{x}) = U_{\text{chiral}}(t, \vec{x}) \vec{\Gamma} (U_{\text{chiral}}(t, \vec{x}))^{-1} \\
\Gamma^5 &\rightarrow (\Gamma^5(t, \vec{x}))' = U_{\text{chiral}}(t, \vec{x}) \Gamma^5 (U_{\text{chiral}}(t, \vec{x}))^{-1}.
\end{aligned} \tag{543}$$

and $\vec{A}(t, \vec{x}) = iU_{\text{chiral}}(t, \vec{x}) \vec{\nabla} (U_{\text{chiral}}(t, \vec{x}))^{-1}$ is an "effective" chiral gauge field.

In the following part, to describe the physics of non-uniform chiral order, we introduce the varied Gamma matrices as

$$\begin{aligned}
(\gamma^0(t, \vec{x}))' &= (\Gamma^5(t, \vec{x}))' = U_{\text{chiral}}(t, \vec{x}) \Gamma^5 (U_{\text{chiral}}(t, \vec{x}))^{-1}, \\
(\gamma^1(t, \vec{x}))' &= U_{\text{chiral}}(t, \vec{x}) \gamma^1 (U_{\text{chiral}}(t, \vec{x}))^{-1}.
\end{aligned} \tag{544}$$

In particular, we point out that the quantum field of varied Gamma matrix $(\gamma^0(t, \vec{x}))'$ may lead to an emergent curved space with varied reference-frame.

D. Chiral-spacetime-crystal – an unbalancedly-rotating knot-crystal with chiral-density-wave-order

1. Chiral-density-wave-order

Chiral-spacetime-crystal is a key concept in chiral-gravity physics that is an unbalancedly-rotating knot-crystal with chiral-density-wave (chiral-DW) order.

Firstly, we consider a 1-level balancedly-rotating (Dirac) knot-crystal with knot-function as

$$\mathbf{Z}_{\text{Dirac}}(\vec{x}, t) = \begin{pmatrix} r_A e^{\pm i\vec{k}_0 \cdot \vec{x} + i\omega_A t + i\phi_{0,A}} \\ r_B e^{\pm i\vec{k}_0 \cdot \vec{x} + i\omega_B t + i\phi_{0,B}} \end{pmatrix}, \tag{545}$$

where $\phi_{0,A/B}$ is a constant angle for A/B-brane. $\vec{k}_0 = \frac{\pi}{a}\vec{e}$, $r_A \simeq r_B = r_0$ is constant, a is a fixed length that denotes the half pitch of windings and \vec{e} is given direction in 3D space. Next, we consider a 1-level $\mathcal{N} = 1$ chiral knot-crystal that is described by

$$\mathbf{Z}_{\text{chiral-space}}(\vec{x}, t) = \begin{pmatrix} r_A \\ r_B \exp(i\vec{K} \cdot \vec{x}) \end{pmatrix} \quad (546)$$

where $\vec{K} = \frac{\pi}{a_0}\vec{e}$ becomes a wave-vector of chiral-density-wave order and $a_0 \gg a$ is the corresponding wave-length.

By combining the 3D 1-level balancedly-rotating knot-crystal $\mathbf{Z}_{\text{Dirac}}(\vec{x}, t)$ and the 1-level $\mathcal{N} = 1$ chiral knot-crystal $\mathbf{Z}_{\text{chiral-space}}(\vec{x}, t)$ together, we get a composite 3D 1-level balancedly-rotating with chiral-density-wave-order $\mathbf{Z}_{\text{chiral-DW}}(\vec{x}, t)$ as

$$\mathbf{Z}_{\text{chiral-DW}}(\vec{x}, t) = \mathbf{Z}_{\text{Dirac}}(\vec{x}, t) \otimes \mathbf{Z}_{\text{chiral-space}}(\vec{x}, t) \quad (547)$$

In physics, a composite 3D 1-level balancedly-rotating is really a 3D 1-level balancedly-rotating with a fixed brane-twisting pattern $\exp(i\vec{K} \cdot \vec{x})$. For this reason, a chiral-density-wave order is a *solidifying motion* of brane-twisting.

2. Chiral-space-crystal and chiral-spacetime-crystal

$\mathbf{Z}_{\text{chiral-space}}(\vec{x}, t)$ denotes a uniform changing of the length-difference between brane-A and brane-B in a knot-crystal. Because the chiral angle $\Theta(\vec{x}, t) = \phi_A(\vec{x}, t) - \phi_B(\vec{x}, t)$ is proportion to spacial coordinate, $\Theta(\vec{x}, t) = \vec{K} \cdot \vec{x}$, we call $\mathbf{Z}_{\text{chiral-space}}(\vec{x}, t)$ to be *chiral-space-crystal* with linear spacial-changing Θ . In mathematic, the definition of a chiral-spacial-crystal is based on spacial-translation operation $\mathcal{T}(\Delta\vec{x})$, i.e.,

$$\begin{aligned} \mathcal{T}(\Delta\vec{x})z_A(\vec{x}, t) &= e^{i\Delta\phi_A}z_A(\vec{x}, t), \\ \mathcal{T}(\Delta\vec{x})z_B(\vec{x}, t) &= e^{i\Delta\phi_B}z_B(\vec{x}, t). \end{aligned} \quad (548)$$

For chiral-spacial-crystal, the chiral angle is obtained as

$$\begin{aligned} \Theta(\vec{x}, t) &= \phi_A(\vec{x}, t) - \phi_B(\vec{x}, t) \\ &= \Delta\phi_A - \Delta\phi_B \\ &= \vec{\alpha} \cdot \Delta\vec{x} \end{aligned} \quad (549)$$

where $\vec{\alpha}$ is a constant vector. We may define the generation operator of chiral-spacial-crystal as

$$\hat{U}(\mathbf{Z}_{\text{chiral-space}}) = \begin{pmatrix} 1 \\ \exp((i\vec{K} \cdot \Delta\vec{x})\hat{K}) \end{pmatrix} \quad (550)$$

where $\hat{K} = -i\frac{d}{d\phi_B}$.

Next, we consider the generalized translation symmetry of $\mathbf{Z}_{\text{chiral-DW}}(\vec{x}, t)$. Due to the existence of the chiral-density-wave order, the continuous translation invariance of $\mathbf{Z}_{\text{chiral-DW}}(\vec{x}, t)$ is broken down to that of $\mathbf{Z}_{\text{chiral-space}}(\vec{x}, t)$. Under a spacial translation operation $\mathcal{T}(\Delta\vec{x})$, we have the generalized translation operation for $\mathbf{Z}_{\text{chiral-DW}}(\vec{x}, t)$ as

$$\begin{aligned} \mathcal{T}(\Delta\vec{x})\mathbf{Z}_{\text{chiral-DW}}(\vec{x}, t) &= [\mathcal{T}(\Delta\vec{x})\mathbf{Z}_{\text{Dirac}}(\vec{x}, t)] \\ &\quad \otimes [\mathcal{T}(\Delta\vec{x})\mathbf{Z}_{\text{chiral-space}}(\vec{x}, t)], \end{aligned} \quad (551)$$

where

$$\mathcal{T}(\Delta\vec{x})\mathbf{Z}_{\text{chiral-space}}(\vec{x}, t) = \begin{pmatrix} 1 \\ \exp[i(\frac{\pi}{a_0}\vec{e} \cdot \Delta\vec{x})] \end{pmatrix} \mathbf{Z}_{\text{chiral-space}}(\vec{x}, t). \quad (552)$$

Then, by considering unbalancedly-rotating knot-crystal with a chiral-density-wave order, we define chiral-spacetime-crystal by combining a chiral-time-crystal and a chiral-space-crystal

$$\begin{aligned} \mathbf{Z}_{\text{chiral-spacetime}}(\vec{x}, t) &= \mathbf{Z}_{\text{chiral-time}}(\vec{x}, t) \otimes \mathbf{Z}_{\text{chiral-space}}(\vec{x}, t) \\ &= \begin{pmatrix} 1 \\ \exp[i(\frac{\pi}{a_0}\vec{e} \cdot \Delta\vec{x})] \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \exp(i\Delta\omega \cdot t) \end{pmatrix} \begin{pmatrix} r_A \\ r_B \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ \exp[i(\frac{\pi}{a_0}\vec{e} \cdot \Delta\vec{x} + \Delta\omega \cdot t)] \end{pmatrix} \begin{pmatrix} r_A \\ r_B \end{pmatrix}. \end{aligned} \quad (553)$$

Thus, for an unbalancedly-rotating knot-crystal with a chiral-density-wave order, the knot-function becomes

$$\begin{aligned}\mathbf{Z}_{\text{chiral-DW}}(\vec{x}, t) &= \mathbf{Z}_{\text{Dirac}}(\vec{x}, t) \otimes \mathbf{Z}_{\text{chiral-spacetime}}(\vec{x}, t) \\ &= \mathbf{Z}_{\text{Dirac}}(\vec{x}, t) \otimes \mathbf{Z}_{\text{chiral-time}}(\vec{x}, t) \\ &\quad \otimes \mathbf{Z}_{\text{chiral-space}}(\vec{x}, t).\end{aligned}\tag{554}$$

According to spacial continuous translation symmetry breaking, the low energy excitations become composite knots that have constant distribution in a unit-cell of chiral-density-wave order.

3. Equivalence between spacetime coordinates and chiral angles

We point out that for chiral-spacetime-crystal the spacial coordinates $\vec{x} = (x, y, z)$ are proportion to the chiral angles $\vec{\Theta} = (\Theta_x, \Theta_y, \Theta_z)$.

To create a chiral-density-wave order, we do a chiral transformation as

$$\begin{aligned}\hat{U}_{0,\text{chiral}}(t, \vec{x}) &= \exp(i\vec{\Theta} \cdot (\tau_z \otimes \vec{\sigma}/2)) \\ &\quad \times \exp(i\Theta_t \cdot \tau_z)\end{aligned}\tag{555}$$

where

$$\vec{\Theta} = \frac{\pi}{a_0} \cdot \vec{x}, \quad \Theta_t = \frac{\Delta\omega}{2} \cdot t.\tag{556}$$

For chiral-spacetime-crystal, the chiral angles are proportion to spacial shift or temporal shift, i.e.,

$$\begin{aligned}\text{Spacetime coordinates } (\vec{x}, t) \\ \iff \text{chiral angles } (\vec{\Theta}, \Theta_t).\end{aligned}\tag{557}$$

In mathematic, this effect comes from spacetime translation operation, i.e.,

$$\begin{aligned}\mathcal{T}(\Delta\vec{x}) \cdot \mathcal{T}(\Delta t) &= \exp(i\vec{\Theta} \cdot (\tau_z \otimes \vec{\sigma}/2)) \\ &\quad \cdot \exp(i\Theta_t \cdot \tau_z).\end{aligned}\tag{558}$$

A spacial translation is realized by doing chiral operation $\exp(i\vec{\Theta} \cdot (\tau_z \otimes \vec{\sigma}/2))$ and a temporal translation is realized by doing chiral operation $\exp(i\Theta_t \cdot \tau_z)$.

In addition, to characterize different types of chiral orders $(\Gamma^a(t, \vec{x}))' = \hat{U}_{0,\text{chiral}}(t, \vec{x})\Gamma^a(\hat{U}_{0,\text{chiral}}(t, \vec{x}))^{-1}$, we had introduced a classical unit vector-field $n^a(t, \vec{x})$ by the relation as

$$(\Gamma^5(t, \vec{x}))' = \sum_a \Gamma^a n^a(t, \vec{x})\tag{559}$$

where $\mathbf{n} = (n^1, n^2, n^3, \phi_0^5) = (\vec{n}, \phi_0^5)$ is an $\text{SO}(4)$ rotor that denotes the direction of chiral order determined by the chiral angle $\Theta = \phi_{0,A} - \phi_{0,B}$ along a given direction \vec{e} .

4. Effective quantized spacetime

In this part, we consider the chiral-spacetime-crystal as an effective "quantized spacetime" for the composite knots.

The coordinate measurement of the effective "spacetime" from chiral-spacetime-crystal is the chiral angles, $(\vec{\Theta}, \Theta_t)$. Along a given direction \vec{e} , after shifting the distance $2a_0$, the chiral angle changes 2π . Thus, the unit-cell of chiral-spacetime-crystal is $2a_0$. The effective "spacetime" from chiral-spacetime-crystal is quantized. The position is determined by two kinds of values: \vec{X} are integer numbers

$$\vec{X} = (X, Y, Z) = \frac{1}{2\pi}\vec{\Theta} - \frac{1}{2\pi}\vec{\Theta} \bmod 2\pi\tag{560}$$

and $\vec{\phi}$ denote internal chiral angles

$$\vec{\phi} = (\phi_x, \phi_y, \phi_z) = \vec{\Theta} \bmod 2\pi\tag{561}$$

with $\phi_x, \phi_y, \phi_z \in (0, 2\pi]$.

Therefore, on the chiral-spacetime-crystal, the effective Hamiltonian turns into

$$\begin{aligned}\hat{h}_{\text{chiral-spacetime}} &= \vec{\Gamma}' \cdot \vec{p} + m (\Gamma^5)' \\ &= \vec{\Gamma}' \cdot \vec{p}_X + \vec{\Gamma}' \cdot \vec{p}_\phi + m (\Gamma^5)'\end{aligned}\quad (562)$$

where $(\Gamma^5)' = \hat{U}_{0,\text{chiral}}(t, \vec{x}) \Gamma^5 (\hat{U}_{0,\text{chiral}}(t, \vec{x}))^{-1}$, $\vec{\Gamma}' = \hat{U}_{0,\text{chiral}}(t, \vec{x}) \vec{\Gamma} (\hat{U}_{0,\text{chiral}}(t, \vec{x}))^{-1}$, $\vec{p}_X = \frac{1}{2a_0} i \frac{d}{dX}$ and $\vec{p}_\phi = \frac{1}{2a_0} i \frac{d}{d\phi}$. The chiral operation for the chiral-spacetime-crystal along a given direction \vec{e} is given by

$$\begin{aligned}\hat{U}_{0,\text{chiral}}(t, \vec{x}) &= \exp(i \vec{\Theta} \cdot (\tau_z \otimes \vec{\sigma}/2)) \\ &\times \exp(i \Theta_t \cdot \tau_z).\end{aligned}\quad (563)$$

Because of $\phi_j \in (0, 2\pi]$, quantum number of \vec{p}_ϕ is angular momentum \vec{L}_ϕ and the energy spectra are $\frac{1}{2a_0} \left| \vec{L}_\phi \right|$. If we focus on the low energy physics $E \ll \frac{1}{2a_0}$ (or $\vec{L}_\phi = 0$), we may get the low energy effective Hamiltonian as

$$\hat{h}_{\text{chiral-spacetime}} \simeq \vec{\Gamma}' \cdot \vec{p}_X + m (\Gamma^5)'. \quad (564)$$

Because of $x_\mu = (\frac{2a_0}{2\pi}) \cdot \Theta_\mu$, the effective action becomes

$$\mathcal{S}_{\text{chiral-spacetime}} \simeq (2a_0)^4 \sum_{X,Y,Z,X_0} \mathcal{L}_{\text{chiral-spacetime}} \quad (565)$$

where

$$\mathcal{L}_{\text{chiral-spacetime}} = \bar{\psi} [i \frac{1}{(\frac{1}{2a_0})} (\gamma^\mu)' \hat{\partial}'_\mu - m] \psi \quad (566)$$

where $(\gamma^1)' = (\gamma^0)'(\Gamma^1)'$, $(\gamma^2)' = (\gamma^0)'\Gamma^2$, $(\gamma^3)' = (\gamma^0)'(\Gamma^3)'$, and $(\gamma^0)' = (\Gamma^5)'$, $\hat{\partial}'_\mu = \frac{\partial}{\partial X_\mu}$.

For low energy physics, the position in 3+1D spacetime is labeled by four integer numbers, $\mathbf{X} = (\vec{X}, X_0)$ where

$$\begin{aligned}\vec{X} &= \frac{1}{2\pi} \vec{\Theta} - \frac{1}{2\pi} \vec{\Theta} \bmod 2\pi, \\ X_0 &= \frac{1}{2\pi} \Theta_t - \frac{1}{2\pi} \Theta_t \bmod 2\pi.\end{aligned}\quad (567)$$

As a result, the effective "spacetime" becomes *quantized*. To characterize its highly nontrivial property, the building block of quantized spacetime is regarded to be *spacetime-chip* with topological structure. Thus, the size of a spacetime-chip is $2a_0$. In a spacetime-chip, the chiral angle changes 2π that corresponds to two composite knots.

A *dimensionless* quantum field theory on a uniform quantized spacetime is obtained to describe low energy physics

$$\mathcal{S}_{\text{chiral-spacetime}} = \sum_{X,Y,Z,X_0} \bar{\Psi} (i \gamma^\mu \hat{\partial}'_\mu - \tilde{m}) \Psi \quad (568)$$

where $\tilde{m} = (2a_0) \cdot m$ is a dimensionless mass and

$$\psi(\mathbf{X}) = (2a_0)^{-3/2} \cdot \Psi(\mathbf{X}) \cdot \hat{U}_{0,\text{chiral}}(\mathbf{X}). \quad (569)$$

The reduced Gamma matrices become $\gamma^1 = \gamma^0 \Gamma^1$, $\gamma^2 = \gamma^0 \Gamma^2$, $\gamma^3 = \gamma^0 \Gamma^3$, and $\gamma^0 = \Gamma^5$. In particular, all parameters, operators and fields in the dimensionless quantum field theory on quantized spacetime (\mathbf{X} , $\hat{\partial}'_\mu$, Ψ , \tilde{m} , ...) are dimensionless!

E. Emergent curved spacetime – deformed chiral-spacetime-crystal

In above section, we pointed out that the coordinates of the effective "spacetime" are the chiral angles, $(\vec{\Theta}, \Theta_t)$. However, because the chiral angles $(\vec{\Theta}, \Theta_t)$ are dynamic degrees of freedom, the fluctuations of chiral-density-wave order lead to an emergent curved spacetime.

1. *Deformed chiral-spacetime-crystal as emergent curved quantized spacetime*

Due to fluctuations, a non-uniform chiral-density-wave order becomes a deformed chiral-spacetime-crystal. In the following parts, we will study the properties of a deformed chiral-spacetime-crystal.

For a non-uniform chiral-density-wave order, we have

$$\begin{aligned}\hat{U}_{0,\text{chiral}}(\vec{\Theta}, \Theta_t) &\implies \hat{U}'_{\text{chiral}}(\vec{\Theta}, \Theta_t) \\ &= \hat{U}_{0,\text{chiral}}(\vec{\Theta}, \Theta_t) \cdot \hat{U}''_{\text{chiral}}(\vec{\Theta}, \Theta_t).\end{aligned}\quad (570)$$

Here $\hat{U}_{0,\text{chiral}}(\vec{\Theta}, \Theta_t)$ denotes the background chiral-density-wave order and $\hat{U}''_{\text{chiral}}(\vec{\Theta}, \Theta_t)$ denotes an additional chiral transformation that leads to a slightly deviation from ground state. Due to the existence $\hat{U}''_{\text{chiral}}(\vec{\Theta}, \Theta_t)$, the deformed chiral-spacetime-crystal with a non-uniform chiral-density-wave order is described by

$$\begin{aligned}(\vec{\Theta}, \Theta_t) \\ \implies (\vec{\Theta}'(\vec{\Theta}, \Theta_t), \Theta'_t(\vec{\Theta}, \Theta_t)).\end{aligned}\quad (571)$$

From the point view of chiral order, $\hat{U}''_{\text{chiral}}(\vec{\Theta}, \Theta_t)$ changes the internal degrees of freedom that is denoted by non-uniform chiral-density-wave order, i.e.,

$$\begin{aligned}\vec{\Gamma} &\rightarrow \vec{\Gamma}'(\vec{\Theta}, \Theta_t) = \hat{U}''_{\text{chiral}}(\vec{\Theta}, \Theta_t) \vec{\Gamma} (\hat{U}''_{\text{chiral}}(\vec{\Theta}, \Theta_t))^{-1} \\ \Gamma^5 &\rightarrow (\Gamma^5(\vec{\Theta}, \Theta_t))' = U''_{\text{chiral}}(\vec{\Theta}, \Theta_t) \Gamma^5 (\hat{U}''_{\text{chiral}}(\vec{\Theta}, \Theta_t))^{-1}\end{aligned}\quad (572)$$

that can be also characterized by varied Gamma matrices

$$\begin{aligned}(\gamma^0(\vec{\Theta}, \Theta_t))' &= (\Gamma^5)'(\vec{\Theta}, \Theta_t) = U_{\text{chiral}}(\vec{\Theta}, \Theta_t) \Gamma^5 (\hat{U}''_{\text{chiral}}(\vec{\Theta}, \Theta_t))^{-1}, \\ (\gamma^1(\vec{\Theta}, \Theta_t))' &= U''_{\text{chiral}}(\vec{\Theta}, \Theta_t) \gamma^1 (\hat{U}''_{\text{chiral}}(\vec{\Theta}, \Theta_t))^{-1}.\end{aligned}\quad (573)$$

From the point view of chiral-spacetime-crystal, $\hat{U}''_{\text{chiral}}(\vec{\Theta}, \Theta_t)$ changes the spacetime degrees of freedom that is denoted by the local coordination transformation, i.e.,

$$\begin{aligned}(\vec{\Theta}, \Theta_t) &\implies (\vec{\Theta}'(\vec{\Theta}, \Theta_t), \Theta'_t(\vec{\Theta}, \Theta_t)) \\ &= (\vec{\Theta} + \delta\vec{\Theta}, \Theta_t + \delta\Theta_t).\end{aligned}\quad (574)$$

This equation just implies a curved spacetime: the lattice constants of the chiral-spacetime-crystal (the size of a spacetime-chip with 2π chiral angle changing) are not fixed to be $2a_0$, rather than $\alpha_i(\vec{\Theta}, \Theta_t) \cdot 2a_0$ where $\alpha_i(\vec{\Theta}, \Theta_t)$ is a scaling factor from $\hat{U}''_{\text{chiral}}(\vec{\Theta}, \Theta_t)$. The distance between two nearest-neighbor "lattice sites" on the spacial coordinate changes, i.e.,

$$\begin{aligned}\Delta\vec{x} &= (\vec{x} + \vec{e}_x) - \vec{x} = \vec{e}_x \implies \\ \Delta\vec{x}' &= (\vec{x}' + \vec{e}'_x) - \vec{x}' = \vec{e}'_x = \alpha_x \vec{e}_x\end{aligned}\quad (575)$$

where \vec{e}_i and \vec{e}'_i are the unit-vectors of the original frame and the deformed frame, respectively. Therefore, we have

$$\begin{aligned}\text{Local chiral transformation } \hat{U}''_{\text{chiral}}(\vec{\Theta}, \Theta_t) \\ \implies \text{Diffeomorphism transformation.}\end{aligned}\quad (576)$$

On the one hand, to fully characterize the changes of chiral angles $(\vec{\Theta}', \Theta'_t)$, we must consider an effective curved quantized spacetime by using a geometry-representation, $\mathbf{x} = (\vec{x}, t) \rightarrow \mathbf{x}' = (\vec{x}', t')$. On the other hand, we need to consider a varied classical field $(\Gamma^5(\mathbf{x}))' = \sum_a \Gamma^a n^a(\mathbf{x})$ by using an effective gauge-field-representation. In next parts, we will develop both geometry-representation and gauge-field-representation and show their relationship. Based on the two representations, we obtain a complete theory of gravitational interaction and find that gravity is a topological chiral-orbital coupling effect.

2. Geometry-representation for deformed chiral-spacetime-crystal

In above section, we have shown an emergent curved spacetime for deformed chiral-spacetime-crystal. The effect of a 3+1D curved spacetime with local $\text{SO}(3,1)$ Lorentz symmetry can be characterized by geometry-representation. The geometry fields (vierbein fields $e^a(\mathbf{x})$ and spin connections $\omega^{ab}(\mathbf{x})$) are determined by the nonuniform non-uniform chiral-density-wave order $(\gamma^\mu(\mathbf{x}))'$. In particular, varied $(\gamma^0(\mathbf{x}))'$ indicates a varied light-core that just means a curved spacetime.

For the case of deformed chiral-spacetime-crystal $(\vec{\Theta}, \Theta_t) \Rightarrow (\vec{\Theta}'(\vec{\Theta}, \Theta_t), \Theta'_t(\vec{\Theta}, \Theta_t))$, the quantized spacetime will never change, i.e.,

$$\begin{aligned}\vec{X} &\rightarrow \vec{X}' = \frac{1}{2\pi}\vec{\Theta} - \frac{1}{2\pi}\vec{\Theta} \bmod 2\pi = \vec{X}, \\ X_0 &\rightarrow X'_0 = \frac{1}{2\pi}\Theta_t - \frac{1}{2\pi}\Theta_t \bmod 2\pi = X_0.\end{aligned}\quad (577)$$

It is obtained to describe low energy physics with non-uniform chiral-density-wave order on an effective quantized spacetime as

$$\mathcal{S}_{\text{chiral-spacetime}} = \sum_{X,Y,Z,X_0} \bar{\Psi}(i\gamma^\mu \hat{\partial}'_\mu - \tilde{m})\Psi. \quad (578)$$

Due to the invariance of a linear knot on quantized spacetime \mathbf{X} (a linear knot is always a half-winding with π chiral angle changing, $\Delta\Theta \equiv \pi$), the deformation of chiral-density-wave order doesn't change the low energy effective model for composite knots of chiral-spacetime-crystal as

$$\begin{aligned}\mathcal{S}_{\text{chiral-spacetime}} &\rightarrow \mathcal{S}'_{\text{chiral-spacetime}} \\ &= \hat{U}''_{\text{chiral}}(\mathbf{X}) \mathcal{S}_{\text{chiral-spacetime}} [\hat{U}''_{\text{chiral}}(\mathbf{X})]^{-1} \\ &= \mathcal{S}_{\text{chiral-spacetime}}\end{aligned}\quad (579)$$

and

$$\begin{aligned}\psi(\mathbf{X}) &\rightarrow \psi'(\mathbf{X}) = \psi(\mathbf{X}) \hat{U}''_{\text{chiral}}(\mathbf{X}) \\ &= (2a_0)^{-3/2} \cdot \Psi(\mathbf{X}) \cdot \hat{U}_{0,\text{chiral}}(\mathbf{X}) \cdot \hat{U}''_{\text{chiral}}(\mathbf{X}).\end{aligned}\quad (580)$$

Since the non-uniform chiral-density-wave order leads to a deformed chiral-spacetime-crystal, we may arbitrarily deform the chiral-spacetime-crystal by doing non-uniform chiral transformation $\hat{U}''_{\text{chiral}}(\mathbf{X})$ and get the same low energy effective model. Thus, we have *diffeomorphism invariance* for the low energy effective model of chiral-spacetime-crystal, i.e.,

$$\begin{aligned}&\text{Model-invariance in quantized} \\ &\text{spacetime under chiral transformation} \\ &\Rightarrow \text{Diffeomorphism invariance.}\end{aligned}\quad (581)$$

As a result, after doing coordinate transformation $\mathbf{X} \rightarrow \mathbf{x} = (\vec{x}, t)$ [35], the low energy physics of deformed chiral-spacetime-crystal on curved spacetime turns into

$$\begin{aligned}\mathcal{S}_{\text{chiral-spacetime}} &= \sum_{X,Y,Z,X_0} \bar{\Psi}(i\gamma^\mu \hat{\partial}'_\mu - \tilde{m})\Psi \\ &= \int \sqrt{-g(x)} \bar{\psi}(e_a^\mu \gamma^a (i\hat{\partial}_\mu + i\omega_\mu) - m)\psi d^4x\end{aligned}\quad (582)$$

where $\omega_\mu = (\omega_\mu^{0i} \gamma^{0i}/2, \omega_\mu^{ij} \gamma^{ij}/2)$ ($i, j = 1, 2, 3$) and $\gamma^{ab} = -\frac{1}{4}[\gamma^a, \gamma^b]$ ($a, b = 0, 1, 2, 3$). This model described by $\mathcal{S}_{\text{chiral-spacetime}}$ is invariant under local $\text{SO}(3,1)$ Lorentz transformation

$$S_{\text{Lor}}(\mathbf{x}) = e^{\Theta_{ab}(\mathbf{x})\gamma^{ab}} \quad (583)$$

as

$$\begin{aligned}\Psi &\rightarrow \Psi' = S_{\text{Lor}}(\mathbf{x})\Psi, \\ \gamma^\mu &\rightarrow (\gamma^\mu(\mathbf{x}))' = S_{\text{Lor}}(\mathbf{x})\gamma^\mu S_{\text{Lor}}^{-1}(\mathbf{x}), \\ \omega_\mu &\rightarrow \omega'_\mu(\mathbf{x}) = S_{\text{Lor}}(\mathbf{x})\omega_\mu(\mathbf{x})S_{\text{Lor}}^{-1}(\mathbf{x}) + S_{\text{Lor}}(\mathbf{x})\partial_\mu S_{\text{Lor}}^{-1}(\mathbf{x}).\end{aligned}\quad (584)$$

γ^5 is invariant under local $\text{SO}(3,1)$ Lorentz symmetry as

$$\gamma^5 \rightarrow (\gamma^5)' = S_{\text{Lor}}(\mathbf{x})\gamma^5 S_{\text{Lor}}^{-1}(\mathbf{x}) = \gamma^5. \quad (585)$$

In addition to the existence of a set of vierbein fields e^a , the space metric is defined by $\eta_{ab}e_\alpha^a e_\beta^b = g_{\alpha\beta}$ where η is the internal space metric tensor. Furthermore, one needs to introduce spin connections $\omega^{ab}(\mathbf{x})$ and the Riemann curvature 2-form as

$$R_b^a = d\omega_b^a + \omega_c^a \wedge \omega_b^c = \frac{1}{2}R_{b\mu\nu}^a d\mathbf{x}^\mu \wedge d\mathbf{x}^\nu, \quad (586)$$

where $R_{b\mu\nu}^a \equiv e_\alpha^a e_b^\beta R_{\beta\mu\nu}^\alpha$ are the components of the usual Riemann tensor projection on the tangent space.

3. Gauge-field-representation for deformed chiral-spacetime-crystal

For the non-uniform chiral-density-wave order on quantized spacetime, there exists an $\text{SO}(4)$ gauge symmetry. We then derive the action in gauge-field-representation on curved quantized spacetime as

$$\begin{aligned} \mathcal{S}_{\text{chiral-spacetime}} = \int \sqrt{-g(x)} \bar{\psi} (e_a^\mu \gamma^a (i\hat{\partial}_\mu + i\omega_\mu \\ + iA_\mu) - m) \psi d^4x \end{aligned} \quad (587)$$

with $A_\mu = A_\mu^{ab}\Gamma^{ab}/2$ ($a, b = 1, 2, 3, 5$) and $\Gamma^{ab} = -\frac{1}{4}[\Gamma^a, \Gamma^b]$ ($a, b = 1, 2, 3, 5$).

This model described by $\mathcal{S}_{\text{chiral-spacetime}}$ is invariant under local $\text{SO}(4)$ chiral transformation

$$U_{\text{SO}(4)}(\mathbf{x}) = e^{\Theta_{ab}(\mathbf{x})\Gamma^{ab}} \quad (588)$$

as

$$\begin{aligned} \Psi &\rightarrow \Psi' = U_{\text{SO}(4)}(\mathbf{x})\Psi, \\ \Gamma^a &\rightarrow (\Gamma^a(\mathbf{x}))' = U_{\text{SO}(4)}(\mathbf{x})\Gamma^a U_{\text{SO}(4)}^{-1}(\mathbf{x}), \\ A_\mu &\rightarrow A'_\mu = U_{\text{SO}(4)}(\mathbf{x})A_\mu U_{\text{SO}(4)}^{-1}(\mathbf{x}) + U_{\text{SO}(4)}(\mathbf{x})\partial_\mu U_{\text{SO}(4)}^{-1}(\mathbf{x}). \end{aligned} \quad (589)$$

The non-uniform chiral-density-wave order is characterized by gauge field strength $F^{ab} = dA^{ab} + A^{ac} \wedge A^{cb}$. In particular, we emphasize that Γ^4 is invariant under local $\text{SO}(4)$ chiral transformation as

$$\Gamma^4 \rightarrow U_{\text{SO}(4)}(\mathbf{x})\Gamma^4 U_{\text{SO}(4)}^{-1}(\mathbf{x}) = \Gamma^4. \quad (590)$$

To characterize chiral order, we introduce the chiral gauge field A^{ab} that can be written into two parts: $\text{SO}(3)$ parts

$$A^{ij} = \text{tr}(\Gamma^{ij} U_{\text{SO}(4)} dU_{\text{SO}(4)}^{-1}(\mathbf{x})) \quad (591)$$

and $\text{SO}(4)/\text{SO}(3)$ parts

$$\begin{aligned} A^{i5} &= \text{tr}(\Gamma^{i5} U_{\text{SO}(4)} dU_{\text{SO}(4)}^{-1}(\mathbf{x})) \\ &= \Gamma^5 d(\Gamma^i(\mathbf{x}))' = -\Gamma^i d(\Gamma^5(\mathbf{x}))'. \end{aligned} \quad (592)$$

The total gauge field strength $\mathcal{F}^{ij}(\mathbf{x})$ of $i, j = 1, 2, 3$ components can be divided into two parts

$$\mathcal{F}^{ij}(\mathbf{x}) = F^{ij} + A^{i5} \wedge A^{j5}. \quad (593)$$

According to pure gauge condition, we have Maurer-Cartan equation,

$$\mathcal{F}^{ij}(\mathbf{x}) = F^{ij} + A^{i5} \wedge A^{j5} \equiv 0 \quad (594)$$

or

$$F^{ij} = dA^{ij} + A^{ik} \wedge A^{kj} \equiv -A^{i5} \wedge A^{j5}. \quad (595)$$

4. Relationship between gauge-field-representation and geometry-representation

In this part, we show the relationship between the gauge-field-representation for non-uniform chiral-density-wave order and the geometry-representation for curved spacetime.

We point out that SO(4) gauge symmetry with local SO(3) spin rotation symmetry and local SO(3,1) Lorentz symmetry with local SO(3) space rotation symmetry have the same sub-group - SO(3) spacial rotation symmetry. The total local symmetry of the system is $\frac{SO(3,1) \otimes SO(4)}{SO(3)}$. As a result, there exists intrinsic relationship between the geometry fields $e^a(\mathbf{x})$, $\omega^{ab}(\mathbf{x})$ and chiral gauge field, A^{ab} ($A^{ij}(\mathbf{x})$, $A^{i5}(\mathbf{x})$).

On the one hand, due to $\gamma^{ij} = \Gamma^{ij}$, according to the definition of $\omega^{ij}(\mathbf{x}) = \frac{1}{2}\text{tr}(\gamma^{ij}\omega(\mathbf{x}))$ and $A^{ij}(\mathbf{x}) = \frac{1}{2}\text{tr}(\Gamma^{ij}A(\mathbf{x}))$, the geometry fields $\omega^{ij}(\mathbf{x})$ and chiral gauge field $A^{ij}(\mathbf{x})$ are really identical as

$$\omega^{ij}(\mathbf{x}) \Longleftrightarrow A^{ij}(\mathbf{x}). \quad (596)$$

On the other hand, we have the relationship between $e^i(\mathbf{x})$ and $A^{i5}(\mathbf{x})$ as

$$e^i(\mathbf{x}) = (2a_0)A^{i5}(\mathbf{x}). \quad (597)$$

Because the relationship $\omega^{ij}(\mathbf{x}) = A^{ij}(\mathbf{x})$ is obvious, we focus on the second one $e^i(\mathbf{x}) = (2a_0)A^{i5}(\mathbf{x})$.

For a non-uniform chiral-density-wave order, we have

$$\begin{aligned} \hat{U}_{0,\text{chiral}}(\vec{\Theta}, \Theta_t) &\Longrightarrow \hat{U}'_{\text{chiral}}(\vec{\Theta}, \Theta_t) \\ &= \hat{U}_{0,\text{chiral}}(\vec{\Theta}, \Theta_t) \cdot \hat{U}''_{\text{chiral}}(\vec{\Theta}, \Theta_t). \end{aligned} \quad (598)$$

Due to the existence of $\hat{U}''_{\text{chiral}}(\vec{\Theta}, \Theta_t)$, the deformed chiral-spacetime-crystal with a non-uniform chiral-density-wave order is given by

$$\begin{aligned} (\vec{\Theta}, \Theta_t) &\Longrightarrow (\vec{\Theta}'(\vec{\Theta}, \Theta_t), \Theta'_t(\vec{\Theta}, \Theta_t)) \\ &= (\vec{\Theta} + \delta\vec{\Theta}, \Theta_t + \delta\Theta_t). \end{aligned} \quad (599)$$

Thus, on chiral-spacetime-crystal, the "lattice distances" become dynamic vector fields $\alpha_i(\mathbf{x}) \cdot 2a_0$. We define the vierbein fields $e^i(\mathbf{x})$ that are supposed to transform homogeneously under the local symmetry, and to behave as ordinary vectors under general coordinate transformations,

$$e^i(\mathbf{x}) = dx^i(\mathbf{x}) = \frac{a_0}{\pi} d\Theta^i(\mathbf{x}). \quad (600)$$

On the other hand, for the smoothly deformed chiral-spacetime-crystal $n^i(\mathbf{x}) \ll 1$, we have

$$\begin{aligned} \frac{d\Theta^i(\mathbf{x})}{2\pi} &= n^i(\mathbf{x}) = \text{tr}[\Gamma^5 d\Gamma^i(\mathbf{x})] \\ &= A^{i5}(\mathbf{x}). \end{aligned} \quad (601)$$

Finally, the relationship between $e^i(\mathbf{x})$ and $A^{i5}(\mathbf{x})$ is obtained as

$$e^i(\mathbf{x}) \equiv (2a_0)A^{i5}(\mathbf{x}). \quad (602)$$

According to the relationship between $e^a(\mathbf{x})$, $\omega^{ab}(\mathbf{x})$ and A^{ab} , the action on curved quantized spacetime becomes

$$\begin{aligned} \mathcal{S}_{\text{chiral-spacetime}} &= \int \sqrt{-g(x)} \bar{\psi} (e_a^\mu \gamma^a (i\hat{\partial}_\mu + i\omega_\mu) - m) \psi d^4x \\ &= \int \sqrt{-g(x)} \bar{\psi} (e_a^\mu \gamma^a (i\hat{\partial}_\mu + i\omega_\mu^{0i} \gamma^{ij}/2 \\ &\quad + \omega_\mu^{ij} \gamma^{ij}/2) - m) \psi d^4x \\ &= \int \sqrt{-g(x)} \bar{\psi} (e_a^\mu \gamma^a (i\hat{\partial}_\mu + i\omega_\mu^{0i} \gamma^{ij}/2 \\ &\quad + A_\mu^{ij} \Gamma^{ij}/2) - m) \psi d^4x \end{aligned} \quad (603)$$

where A_μ^{ij} is gauge field of local SO(4) gauge symmetry and $\omega_\mu = \omega_\mu^{ab} \gamma^{ab}/2$ is the spin connection of local SO(3,1) Lorentz symmetry.

F. Emergent chiral-gravity as topological chiral-orbital coupling effect

In this section, we point out that gravity in knot physics is really a *topological chiral-orbital coupling effect*: on the one hand, the deformation of the chiral-density-wave-order (or chiral-spacetime-crystal) leads to the changes of knot-motions that can be denoted by curved spacetime; on the other hand, the linear knots trapping "magnetic" charges deform the chiral-density-wave-order (chiral-spacetime-crystal) that indicates matter may curve spacetime. Because the gravitational interaction in this paper comes from topological chiral-orbital coupling effect, rather than exchanging spin-2 gapless gravitons, we call the emergent gravity to be *chiral-gravity* and the corresponding gravity theory to be *chiral-gravity theory*.

1. Topological properties of classical linear knot on knot-crystal

In this section we will show that each 3D classical composite knot traps chiral magnetic monopole of SO(4) gauge field and becomes topological defect of chiral-spacetime-crystal.

a. 3D classical linear knot as magnetic monopole Firstly, we review the concept of a classical linear knot. A classical linear knot is a special entanglement between two branes (d -brane-A and d -brane-B) along a given direction \vec{e} , of which the knot-function is given by

$$\mathbf{Z}_{\text{knot,A}}(\vec{x}, t) = \begin{pmatrix} z_A(\vec{x}, t) \\ z_B(\vec{x}, t) \end{pmatrix}_{\text{knot}} = \begin{pmatrix} r_A \\ r_B \exp[i\phi_{\text{knot,B}}(x)] \end{pmatrix} \quad (604)$$

and

$$\mathbf{Z}_{\text{knot,B}}(\vec{x}, t) = \begin{pmatrix} z_A(\vec{x}, t) \\ z_B(\vec{x}, t) \end{pmatrix}_{\text{knot}} = \begin{pmatrix} r_A \exp[i\phi_{\text{knot,A}}(x)] \\ r_B \end{pmatrix} \quad (605)$$

where

$$\phi_{\text{knot,A/B}}(x'_1) = \begin{cases} \phi_0 - \frac{\pi}{2}, & x'_1 \in (-\infty, x_0] \\ \phi_0 - \frac{\pi}{2} \pm k_0(x'_1 - x_0), & x'_1 \in (x_0, x_0 + a] \\ \phi_0 + \frac{\pi}{2}, & x'_1 \in (x_0 + a, \infty) \end{cases}. \quad (606)$$

$x'_1 = \vec{x} \cdot \vec{e}$ is the coordination on the axis along a given direction \vec{e} and ϕ_0 is a constant angle, $k_0 = \frac{\pi}{a}$.

For a 3D classical linear knot for two 3-branes (3-brane-A and 3-brane-B) in 5D $\{x, y, z, \xi, \eta\}$ space, there exist two types of configurations at a given point (\pm in above equation) that are described by two-component (spinor) classical complex fields, $z_{1,A/B}$ and $z_{2,A/B}$. $z_{1,A/B}$ is knot-function of clockwise winding knot and $z_{2,A/B}$ is knot-function of anti-clockwise winding. We use $\mathbf{z} = \begin{pmatrix} z_{1,A/B}(\vec{r})/r_{A/B} \\ z_{2,A/B}(\vec{r})/r_{A/B} \end{pmatrix}$ to characterize the complex phase of a 3D classical linear knot. We may use O(3) rotor-representation to denote a 3D classical linear knot, $N^a(x)\sigma^a = \bar{\mathbf{z}}\sigma\mathbf{z}$ ($a = 1, 2, 3$),

$$\begin{aligned} \mathbf{N}(\vec{r}) &= \frac{\vec{r}}{|\vec{r}|}, \quad r = |\vec{r}| \geq a, \\ \mathbf{N}(\vec{r}) &= \left(\frac{\vec{r}}{|\vec{r}|}\right)^{\frac{|\vec{r}|}{a}}, \quad r = |\vec{r}| < a. \end{aligned} \quad (607)$$

As a result, a 3D knot is really a "magnetic" monopole with an integer topological number as

$$\mathcal{N}_{3D\text{-knot}} = \frac{1}{2\pi} \iint_S \vec{B}(\vec{r}) \cdot d\vec{S} = \pm 1 \quad (608)$$

here $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{a}(\vec{r})$, $a_\mu(\vec{r}) = i\bar{\mathbf{z}}(\vec{r})\partial_\mu\mathbf{z}(\vec{r})$, and S is a closed surface surrounding the knot.

b. 3D classical linear knot as chiral magnetic monopole on a double-helix knot-crystal Secondly, we consider a classical linear knot on a (1-level) double-helix knot-crystal. Now, the knot-function becomes

$$\begin{aligned} \mathbf{Z}_{\text{total}}(\vec{x}, t) &= \begin{pmatrix} z_A(\vec{x}, t) \\ z_B(\vec{x}, t) \end{pmatrix}_{\text{total}} \\ &= \mathbf{U}_{\text{knot}} \cdot \mathbf{Z}_{\text{LL}}(\vec{x}, t) \end{aligned} \quad (609)$$

where

$$\mathbf{U}_{\text{knot}} = \exp[i\phi_{\text{knot,A/B}}(x'_1) \cdot \hat{K}], \quad (610)$$

$$\phi_{\text{knot,A/B}}(x'_1) = \begin{cases} -\frac{\pi}{2}, & x'_1 \in (-\infty, x_0] \\ -\frac{\pi}{2} \pm k_0(x'_1 - x_0), & x'_1 \in (x_0, x_0 + a] \\ \frac{\pi}{2}, & x'_1 \in (x_0 + a, \infty) \end{cases},$$

where $\hat{K} = -i\frac{d}{d\phi_{\text{A/B}}}$ is knot-number operator and $x'_1 = \vec{x} \cdot \vec{e}$ is variable along \vec{e} direction.

From this knot-function of a classical linear knot on a knot-crystal, we find that the chiral angle is changed as $\delta\Theta_i = \pi$. That means a 3D knot leads to a changing of chiral angle in chiral order along arbitrary given direction \vec{e} , i.e.,

$$\mathbf{U}_{\text{knot}} \rightarrow \hat{U}_{\text{chiral}} = \exp(i\pi\tau_z \otimes \vec{\sigma}). \quad (611)$$

Due to the spacial rotation symmetry, the classical linear knot becomes chiral magnetic monopole, i.e.,

$$\begin{aligned} \text{A classical linear knot} &\rightarrow \text{chiral magnetic monopole} \\ &\text{that switches chiral order along} \\ &\text{an arbitrary direction } \vec{e}. \end{aligned} \quad (612)$$

According to above discussion, we find that each knot traps a "magnetic charge" of the SO(4) gauge field, i.e.

$$N_F = \int \sqrt{-g} \psi^\dagger \psi d^3x = q_m \quad (613)$$

where q_m is the "magnetic" charge of SO(4) gauge field. For single knot $N_F = 1$, the "magnetic" charge is $q_m = 1$.

The situation is similar to the "magnetic" monopole from π -phase changing

$$\begin{aligned} \text{A classical linear knot} &\rightarrow \text{magnetic monopole} \\ &\text{that switches phase angle} \\ &\text{along an arbitrary direction } \vec{e}. \end{aligned} \quad (614)$$

c. Chiral defect of a classical linear-knot on chiral-spacetime-crystal Thirdly, we consider a classical linear knot on composite 3D 1-level balancedly-rotating with chiral-density-wave-order with knot-functions

$$\mathbf{Z}_{\text{chiral-DW}}(\vec{x}, t) = \mathbf{Z}_{\text{Dirac}}(\vec{x}, t) \otimes \mathbf{Z}_{\text{chiral-space}}(\vec{x}, t) \quad (615)$$

where

$$\mathbf{Z}_{\text{Dirac}}(\vec{x}, t) = \begin{pmatrix} r_A e^{\pm i\vec{k}_0 \cdot \vec{x} + i\omega_A t + i\phi_{0,A}} \\ r_B e^{\pm i\vec{k}_0 \cdot \vec{x} + i\omega_B t + i\phi_{0,B}} \end{pmatrix} \quad (616)$$

describes a 3D 1-level balancedly-rotating knot-crystal with uniform chiral order and

$$\mathbf{Z}_{\text{chiral-space}}(\vec{x}, t) = \begin{pmatrix} r_A \\ r_B \exp(i\vec{K} \cdot \vec{x} + i\omega t) \end{pmatrix} \quad (617)$$

describes a 1-level $\mathcal{N} = 1$ chiral knot-crystal.

After considering an addition classical linear knot, we have the total knot-function to be

$$\mathbf{Z}_{\text{total}}(\vec{x}, t) = \mathbf{U}_{\text{knot}}[\mathbf{Z}_{\text{chiral-DW}}(\vec{x}, t)].$$

For a composite classical linear knot with uniform distribution on the unit-cell of chiral-spacetime-crystal, we have the total knot-function to be

$$\begin{aligned} \mathbf{Z}'_{\text{total}}(\vec{x}, t) &= \mathbf{Z}_{\text{knot}}(\vec{x}, t) \\ &\otimes \mathbf{U}_{\text{knot}}[\mathbf{Z}_{\text{chiral-space}}(\vec{x}, t)]. \end{aligned} \quad (618)$$

There are two physics consequences of the classical linear knot on the chiral-spacetime-crystal: inducing chiral magnetic monopole for chiral-density-wave order and inducing topological-lattice defect for chiral-spacetime-crystal.

On the one hand, the composite classical linear knot leads to changing of chiral angle in chiral-density-wave order along arbitrary direction \vec{e} , i.e.,

$$\mathbf{U}_{\text{knot}} \rightarrow \hat{U}_{\text{chiral}} = \exp(i\pi\tau_z \otimes \vec{\sigma}). \quad (619)$$

As a result, a composite classical linear knot also traps a "magnetic charge" of the $\text{SO}(4)$ gauge field as

$$\int \sqrt{-g} \psi^\dagger \psi d^3x = q_m \quad (620)$$

where q_m is the "magnetic" charge of $\text{SO}(4)$ gauge field. To characterize the induced magnetic charge, we write down the following constraint

$$\iiint \rho_F dV = \frac{1}{4\pi} \iint \epsilon_{jk}^i F_{jk}^{jk} \cdot dS_i \quad (621)$$

where $F^{jk} = dA^{jk} + A^{ji} \wedge A^{ik} \equiv -A^{j5} \wedge A^{k5}$ and $\rho_F = \sqrt{-g} \psi^\dagger \psi$.

On the other hand, due to $\delta\Theta_i = \pi$ ($i = x, y, z$), the composite classical linear knot removes a "lattice site" of the chiral-spacetime-crystal according to

$$\begin{aligned} \mathbf{U}_{\text{knot}} \rightarrow \Delta L_i &= L_{i,A} - L_{i,B} = \frac{a_0}{\pi} \int d\Theta_i \\ &= \pm a_0. \end{aligned} \quad (622)$$

Thus, along an arbitrary direction, the chiral-spacetime-crystal is deformed.

In next section, we point out that the induced (nonlocal) topological structure of chiral-order around classical linear knots leads to gravity.

2. Einstein-Hilbert action as induced topological mutual BF term

The action on curved quantized spacetime is obtained as

$$\mathcal{S}_{\text{chiral-spacetime}} = \int \sqrt{-g(x)} \bar{\psi} (e_a^\mu \gamma^a \hat{D}_\mu - m) \psi d^4x \quad (623)$$

where

$$\begin{aligned} \hat{D}_\mu &= i\hat{\partial}_\mu + i\omega_\mu \\ &= i\hat{\partial}_\mu + i\omega_\mu^{0i} \gamma^{ij} / 2 + A_\mu^{ij} \Gamma^{ij} / 2. \end{aligned} \quad (624)$$

$\omega_\mu = \omega_\mu^{ab} \gamma^{ab} / 2$ is the spin connection of local $\text{SO}(3,1)$ Lorentz symmetry and A_μ^{ij} is gauge field of local $\text{SO}(4)$ gauge symmetry which is equal to ω_μ^{ij} . We point out that due to chiral-orbital coupling, $\text{SO}(4)$ gauge symmetry and local $\text{SO}(3,1)$ Lorentz symmetry have the same sub-group - $\text{SO}(3)$ spacial rotation symmetry.

We then calculate the induced topological mutual BF term. The topological constraint

$$\iiint \rho_F dV = \frac{1}{4\pi} \iint \epsilon_{jk}^i F_{jk}^{jk} \cdot dS_i \quad (625)$$

can be re-written into

$$i\sqrt{-g} \bar{\psi} \gamma^i (\gamma^{0i} / 2) \psi = \epsilon_{0ijk} \epsilon_{0ijk} \frac{1}{4\pi} \hat{D}_0 F_{jk}^{jk} \quad (626)$$

where $\hat{D}_\mu = i\hat{\partial}_\mu + i\omega_\mu$. We then get

$$i\sqrt{-g} \bar{\psi} \omega_i^{0i} \gamma^i (\gamma^{0i} / 2) \psi = i\epsilon_{0ijk} \epsilon_{0ijk} \omega_i^{0i} \frac{1}{4\pi} \hat{D}_0 F_{jk}^{jk}. \quad (627)$$

In the path-integral formulation, to enforce such topological constraint, we may add a topological mutual BF term \mathcal{S}_{MBF} in the action that is

$$\mathcal{S}_{\text{MBF}} = -\frac{1}{4\pi} \int \epsilon_{0ijk} \epsilon_{\mu\nu\lambda\kappa} R_{\mu\nu}^{0i} F_{\lambda\kappa}^{jk} d^4x.$$

The induced topological mutual BF term S_{MBF} is linear in the conventional strength in R^{0i} and F^{jk} as

$$\begin{aligned} S_{\text{MBF}} &= -\frac{1}{4\pi} \int \epsilon_{0ijk} \epsilon_{\mu\nu jk} R_{\mu\nu}^{0i} F_{jk}^{0i} d^4x \\ &= -\frac{1}{4\pi} \int \epsilon_{0ijk} R^{0i} \wedge F^{jk} \end{aligned} \quad (628)$$

where $R^{0i} = d\omega^{0i} + \omega^{0j} \wedge \omega^{ji}$ and $F^{jk} = dA^{jk} + A^{jl} \wedge A^{lk}$. From $F^{jk} \equiv -A^{j5} \wedge A^{k5}$ and $e^i \wedge e^j = (2a_0)^2 A^{i5} \wedge A^{j5}$. This term is proportional to tetradic Palatini action

$$\begin{aligned} S_{\text{MBF}} &= -\frac{1}{4\pi} \int \epsilon_{0ijk} R^{0i} \wedge F^{jk} \\ &= \frac{1}{4\pi} \int \epsilon_{ijk} R^{0i} \wedge (A^{j5} \wedge A^{k5}) \\ &= \frac{1}{4\pi(2a_0)^2} \int \epsilon_{ijk} R^{0i} \wedge e^j \wedge e^k. \end{aligned} \quad (629)$$

By restoring the local $\text{SO}(3,1)$ Lorentz invariance, the topological mutual BF term S_{MBF} turns into the Einstein-Hilbert action S_{EH} as

$$\begin{aligned} S_{\text{MBF}} &= \frac{1}{4\pi(2a_0)^2} \int \epsilon_{ijk} R^{0i} \wedge e^j \wedge e^k \\ \implies S_{\text{EH}} &= \frac{1}{16\pi(a_0)^2} \int \epsilon_{ijkl} R^{ij} \wedge e^k \wedge e^l \\ &= \frac{1}{16\pi G} \int \sqrt{-g} R d^4x \end{aligned} \quad (630)$$

where G is the induced Newton constant which is $G = a_0^2$. The role of the Planck length is played by $l_p = a_0$, that is the size of a composite knot in chiral-spacetime-crystal. Eq.(630) is constructed from the curvature of the spin connection and the vierbein which is invariant under local Lorentz transformations and whose definition is independent of the choice of background metric.

From above discussion, we derived a low energy effective theory of on chiral-spacetime-crystal as

$$\begin{aligned} S &= S_{\text{chiral-spacetime}} + S_{\text{EH}} \\ &= \int \sqrt{-g(x)} \bar{\psi} (e_a^\mu \gamma^a \hat{D}_\mu - m) \psi d^4x \\ &\quad + \frac{1}{16\pi G} \int \sqrt{-g} R d^4x \end{aligned} \quad (631)$$

where $\hat{D}_\mu = i\partial_\mu + i\omega_\mu$. The variation of the action S via the metric $\delta g_{\mu\nu}$ gives the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \quad (632)$$

As a result, for a chiral-spacetime-crystal, the non-uniform chiral order becomes a background like smooth manifold with emergent diffeomorphism invariance, of which the effective theory turns into *topological* gravity theory. The gravitational interaction between two knots with same "magnetic" charge q_m is much different from the electromagnetic interaction between two electrons in quantum electrodynamics (QED). In chiral-gravity from knot physics, the gravitational force is an indirect interaction by deforming chiral-density-wave order; in QED, the electromagnetic force is a direct interaction by exchanging the internal-winding degrees of freedom. For this reason, although two knots have same "magnetic" charge q_m , the interaction between them is attractive.

3. Topological chiral-orbital coupling effect

For chiral-gravity, an important property is topological chiral-orbital coupling effect that comes from the relationship between non-uniform chiral order and curved spacetime. From the Einstein-Hilbert action $S_{\text{EH}} = S_{\text{MBF}} =$

$-\frac{1}{4\pi} \int \epsilon_{0ijk} R^{0i} \wedge F^{jk}$, we found that the key property is duality between Riemann curvature R^{0i} and strength of gauge field of non-uniform chiral order F^{jk} : finite F^{jk} always leads to finite R^{0i} .

To show the topological chiral-orbital coupling effect, we vary the action S via the spin connections ω_μ^{ab} and gauge field A_μ^{ij} to get the equations of motion.

On the one hand, by vary the action S via the spin connections ω_μ^{0i} , the equation of motion becomes

$$i\sqrt{-g}\bar{\psi}\gamma^\mu(\gamma^{0i}/2)\psi = \epsilon_{0ijk}\epsilon_{\mu\nu\lambda\kappa}\frac{1}{4\pi}\hat{D}_\nu F_{\lambda\kappa}^{jk}. \quad (633)$$

For the case of $\mu = i$, the equation of motion leads to

$$i\sqrt{-g}\bar{\psi}\gamma^i(\gamma^{0i}/2)\psi = \epsilon_{0ijk}\epsilon_{0i\lambda\kappa}\frac{1}{4\pi}\hat{D}_0 F_{\lambda\kappa}^{jk} \quad (634)$$

that is just Eq.627. As a result, a composite classical linear knot traps a "magnetic charge" of the chiral gauge field as

$$\int \sqrt{-g}\psi^\dagger\psi d^3x = q_m \quad (635)$$

where q_m is the "magnetic" charge of chiral gauge field. For the case of $\mu = l \neq i, 0$, the equation of motion leads to

$$\begin{aligned} i\sqrt{-g}\epsilon_{li}\bar{\psi}\gamma^l(\gamma^{0i}/2)\psi &= i\sqrt{-g}\epsilon_{li}\psi^\dagger(\gamma^{li}/2)\psi \\ &= \epsilon_{0ijk}\epsilon_{0l\lambda\kappa}\epsilon_{li}\frac{1}{4\pi}\hat{D}_l F_{\lambda\kappa}^{jk}. \end{aligned} \quad (636)$$

This equation shows the relationship between spin density of knots $\psi^\dagger (S^{li}/2) \psi$ ($S^{li}/2 = \gamma^l\gamma^{0i}/2$) and the current of chiral gauge field. For the case of $\mu = 0$, the equation of motion leads to

$$i\sqrt{-g}\bar{\psi}\gamma^0(\gamma^{0i}/2)\psi = \epsilon_{0ijk}\epsilon_{0i\lambda\kappa}\frac{1}{4\pi}\hat{D}_i F_{\lambda\kappa}^{jk}. \quad (637)$$

This equation shows the relationship between chiral current density of knots $\bar{\psi}(\gamma^i/2)\psi = \psi^\dagger(\Gamma^i/2)\psi$ and the current of chiral gauge field.

By varying the action S via the spin connections A_0^{ij} , the equation of motion becomes

$$i\sqrt{-g}\bar{\psi}\gamma^\mu(\Gamma^{jk}/2)\psi = \epsilon_{0ijk}\epsilon_{\mu\nu\lambda\kappa}\frac{1}{4\pi}\hat{D}_\nu R_{\lambda\kappa}^{0i}. \quad (638)$$

For the case of $\mu = 0$, the equation of motion leads to

$$i\sqrt{-g}\bar{\psi}\gamma^0(\Gamma^{jk}/2)\psi = \epsilon_{0ijk}\epsilon_{0\nu\lambda\kappa}\frac{1}{4\pi}\hat{D}_\nu R_{\lambda\kappa}^{0i}. \quad (639)$$

This equation shows the relationship between spin density of knots $\bar{\psi}\gamma^0(\Gamma^{jk}/2)\psi = \psi^\dagger(\gamma^{jk}/2)\psi$ and spacial changing of the Riemann curvature. For the case of $\mu = j$, the equation of motion leads to

$$i\sqrt{-g}\bar{\psi}\gamma^j(\Gamma^{jk}/2)\psi = \epsilon_{0ijk}\epsilon_{j\nu\lambda\kappa}\frac{1}{4\pi}\hat{D}_\nu R_{\lambda\kappa}^{0i}. \quad (640)$$

This equation shows the relationship between chiral-current density of knots $\bar{\psi}\gamma^j(\Gamma^{jk}/2)\psi = \psi^\dagger\Gamma^k/2\psi$ and the changing of the Riemann curvature. For the case of $\mu = i$, the equation of motion leads to

$$i\sqrt{-g}\epsilon_{ijk}\bar{\psi}\gamma^i(\Gamma^{jk}/2)\psi = \epsilon_{0ijk}\epsilon_{i\nu\lambda\kappa}\frac{1}{4\pi}\hat{D}_\nu R_{\lambda\kappa}^{0i}. \quad (641)$$

This equation shows the relationship between chiral density of knots $\bar{\psi}\gamma^i(\Gamma^{jk}/2)\psi = \psi^\dagger\gamma^5/2\psi$ and the changing of the Riemann curvature.

As a result, we call the topological interplay effect between chiral order and orbital motion to be *topological chiral-orbital coupling effect*.

G. Chiral-gravity in the standard knot-crystal

Our universe is the standard knot-crystal, a three dimensional (3D) 3-level isotropic balancedly-rotating knot-crystal ($\mathcal{N} = 3$), of which the hierarchy series is

$$\begin{pmatrix} \{3.5, N\} \\ \{3.5, N + \delta\} \end{pmatrix} \quad (642)$$

where N is a very large number, $N \gg 1$ (for example, $N \sim 10^{15}$) and δ is a very tiny number, $\delta \ll 1$ (for example, $\delta \sim 10^{-10}$). The knot-function of the standard knot-crystal is defined to be

$$\mathbf{Z}_{\text{standard}}(\vec{x}, t) = \begin{pmatrix} z_{A,1}(\vec{x}, t) & z_{A,2}(\vec{x}, t) & z_{A,3}(\vec{x}, t) \\ z_{B,1}(\vec{x}, t) & z_{B,2}(\vec{x}, t) & z_{B,3}(\vec{x}, t) \end{pmatrix} \quad (643)$$

where

$$\begin{pmatrix} z_{A,3}(\vec{x}, t) \\ z_{B,3}(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} r_{A,3} e^{i\phi_{3,A}(\vec{x}, t)} \\ r_{B,3} e^{i\phi_{3,B}(\vec{x}, t)} \end{pmatrix}, \quad (644)$$

$$\begin{aligned} \left(\frac{z_{A,3}(\vec{x}, t)}{r_{A,3}} \right)^{(\Delta_{23})_A} &= \frac{z_{A,2}(\vec{x}, t)}{r_{A,2}}, \\ \left(\frac{z_{B,3}(\vec{x}, t)}{r_{B,3}} \right)^{(\Delta_{23})_B} &= \frac{z_{B,2}(\vec{x}, t)}{r_{B,2}}, \end{aligned} \quad (645)$$

and

$$\begin{aligned} \left(\frac{z_{A,2}(\vec{x}, t)}{r_{A,2}} \right)^{(\Delta_{12})_A} &= \frac{z_{A,1}(\vec{x}, t)}{r_{A,1}}, \\ \left(\frac{z_{B,2}(\vec{x}, t)}{r_{B,2}} \right)^{(\Delta_{12})_B} &= \frac{z_{B,1}(\vec{x}, t)}{r_{B,1}}, \end{aligned} \quad (646)$$

where

$$\begin{aligned} \phi_{A,3}(\vec{x}, t) &= \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \omega_A t, \\ \phi_{B,3}(\vec{x}, t) &= \frac{\pi}{a}(\vec{e} \cdot \vec{x}) - \omega_B t, \end{aligned} \quad (647)$$

and

$$\begin{aligned} (\Delta_{12})_A &= (\Delta_{12})_B = 3.5, \\ (\Delta_{23})_A &= (\Delta_{23})_B + \delta \gg |(\Delta_{12})_{A/B}|. \end{aligned}$$

The low energy effective theory of the standard knot-crystal is the Standard model in particle physics – an $\text{SU}_{\text{Strong}}(3) \otimes \text{SU}_{\text{Weak}}(2) \otimes \text{U}_Y(1)$ gauge theory with Higgs mechanism. For the standard knot-crystal, the fluctuating rotating velocity of a standard knot-crystal $\omega(\vec{x}, t)$ plays the role of Higgs field $\Phi(\vec{x}, t)$.

An important property of the standard knot-crystal $\mathbf{Z}_{\text{standard}}(\vec{x}, t)$ is chirality that comes from the mismatch between two branes, i.e., $(\Delta_{23})_A - (\Delta_{23})_B = \delta$. Finite δ is a finite filling density of right-hand neutrinos and thus becomes the origin of chirality for neutrinos and weak interaction.

On the other hand, the standard knot-crystal $\mathbf{Z}_{\text{standard}}(\vec{x}, t)$ can be regarded as a combination of a symmetric knot-crystal $\mathbf{Z}_{\text{Dirac}}(\vec{x}, t)$ and a 1-level $\mathcal{N} = 1$ chiral knot-crystal $\mathbf{Z}_{\text{chiral-space}}(\vec{x}, t)$. The symmetric knot-crystal $\mathbf{Z}_{\text{Dirac}}(\vec{x}, t)$ is defined by

$$\mathbf{Z}_{\text{Dirac}}(\vec{x}, t) = \begin{pmatrix} z_{\text{Dirac},A,1}(\vec{x}, t) & z_{\text{Dirac},A,2}(\vec{x}, t) & z_{\text{Dirac},A,3}(\vec{x}, t) \\ z_{\text{Dirac},B,1}(\vec{x}, t) & z_{\text{Dirac},B,2}(\vec{x}, t) & z_{\text{Dirac},B,3}(\vec{x}, t) \end{pmatrix} \quad (648)$$

where

$$\begin{pmatrix} z_{\text{Dirac},3,A}(\vec{x}, t) \\ z_{\text{Dirac},3,B}(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} r_{\text{Dirac},3,A} e^{i\phi_{3,A}(\vec{x}, t)} \\ r_{\text{Dirac},3,B} e^{i\phi_{3,B}(\vec{x}, t)} \end{pmatrix}, \quad (649)$$

$$\begin{aligned} \left(\frac{z_{\text{Dirac},2,A}(\vec{x}, t)}{r_{2,A}} \right)^{(\Delta_{23})_A} &= \frac{z_{\text{Dirac},3,A}(\vec{x}, t)}{r_{3,A}}, \\ \left(\frac{z_{\text{Dirac},2,B}(\vec{x}, t)}{r_{2,B}} \right)^{(\Delta_{23})_B} &= \frac{z_{\text{Dirac},3,B}(\vec{x}, t)}{r_{3,B}}, \end{aligned} \quad (650)$$

and

$$\begin{aligned} \left(\frac{z_{\text{Dirac},1,A}(\vec{x}, t)}{r_{1,A}} \right)^{(\Delta_{12})_A} &= \frac{z_{\text{Dirac},2,A}(\vec{x}, t)}{r_{2,A}}, \\ \left(\frac{z_{\text{Dirac},1,B}(\vec{x}, t)}{r_{1,B}} \right)^{(\Delta_{12})_B} &= \frac{z_{\text{Dirac},2,B}(\vec{x}, t)}{r_{2,B}}. \end{aligned} \quad (651)$$

with

$$\begin{aligned} \phi_{3,A}(\vec{x}, t) &= \frac{\pi}{a}(\vec{e} \cdot \vec{x}) + \omega_0 t, \\ \phi_{3,B}(\vec{x}, t) &= \frac{\pi}{a}(\vec{e} \cdot \vec{x}) - \omega_0 t, \\ (\Delta_{12})_A &= (\Delta_{12})_B = 3.5, \\ (\Delta_{23})_A &= (\Delta_{23})_B \gg |(\Delta_{12})_{A/B}|; \end{aligned} \quad (652)$$

The 1-level $\mathcal{N} = 1$ chiral knot-crystal $\mathbf{Z}_{\text{chiral-space}}(\vec{x}, t)$ is defined by

$$\mathbf{Z}_{\text{chiral-space}}(\vec{e}, \vec{x}, t) = \begin{pmatrix} r_{1,A} & r_{2,A} & r_{3,A} \\ r_{1,B} & r_{2,B} & r_{3,B} \exp(i \frac{\pi \delta}{a}(\vec{e} \cdot \vec{x})) \end{pmatrix}. \quad (653)$$

The length of unit-cell of chiral-density-wave order $2a_0$ is twice of the distance between $N = \frac{1}{\delta}$ nearest-neighbor big-rings a , i.e., $2a_0 = 2Na$ or $a_0 = \frac{a}{\delta}$.

The combination of the standard knot-crystal $\mathbf{Z}_{\text{standard}}(\vec{x}, t)$ by the symmetric knot-crystal $\mathbf{Z}'(\vec{x}, t)$ and the chiral-spacial-crystal $\mathbf{Z}_{\text{chiral-space}}(\vec{x}, t)$ can also be represented by

$$\mathbf{Z}_{\text{standard}} = \mathbf{Z}_{\text{Dirac}} \circledast \mathbf{Z}_{\text{chiral-space}}. \quad (654)$$

or

$$\hat{U}(\mathbf{Z}_{\text{standard}}) = \hat{U}(\mathbf{Z}_{\text{Dirac}}) \cdot \hat{U}(\mathbf{Z}_{\text{chiral-space}}). \quad (655)$$

The knot-crystal-operation of chiral-spacial-crystal $\mathbf{Z}_{\text{chiral-space}}(\vec{x}, t)$ is obtained as

$$\hat{U}(\mathbf{Z}_{\text{chiral-space}}) = \begin{pmatrix} r_{1,A} & r_{2,A} & r_{3,A} \\ r_{1,B} & r_{2,B} & r_{3,B} \exp(i\phi(\vec{x}, t) \cdot \hat{K}) \end{pmatrix} \quad (656)$$

where $\hat{K} = -i \frac{d}{d\phi_{3,B}}$ and $\phi(\vec{x}, t) = \frac{\pi}{a_0}(\vec{e} \cdot \vec{x})$. For the standard knot-crystal, due to $(\Delta_{12})_A - (\Delta_{12})_B = \delta$, there exists a chiral-density-wave order that corresponds to chiral-spacetime-crystal $\mathbf{Z}_{\text{chiral-space}}$.

Thus, we may use similar formula to obtain chiral-gravity theory for the standard knot-crystal. The deformed chiral-spacetime-crystal is described by an emergent curved spacetime. All composite linear knots will trap chiral magnetic monopole of chiral-density-wave order and topological lattice defect of chiral-spacetime-crystal. There exists topological chiral-orbital coupling effect for the standard knot-crystal. As a result, the low energy effective theory of the standard knot-crystal is described by

$$\begin{aligned} S &= S_{\text{SM}} + S_{\text{EH}} \\ &= \mathcal{L}_{\text{SM}} + \frac{1}{16\pi G} \int \sqrt{-g} R d^4x \end{aligned} \quad (657)$$

where

$$\begin{aligned} \mathcal{L}_{\text{SM}} &= \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{Y}}(\text{U}_{\text{Y}}(1)) + \mathcal{L}_{\text{strong}}(\text{SU}_{\text{Strong}}(3)) \\ &\quad + \mathcal{L}_{\text{weak}}(\text{SU}_{\text{weak}}(2)) + \mathcal{L}_{\text{Higgs}}. \end{aligned} \quad (658)$$

All the terms in \mathcal{L}_{SM} had been given in last section. And, for the standard knot-crystal, the Planck length is given by $l_p = a_0$. So, we confirm the fact that *our universe is a standard knot-crystal*.

In addition, we may slightly *tune* the standard knot-crystal and consider a symmetric 3-level double-helix knot-crystal with the hierarchy series

$$\begin{pmatrix} \{3.5, N\} \\ \{3.5, N\} \end{pmatrix} \quad (659)$$

(N is a large number). It was known that the low energy effective theory is an $\text{SU}_{\text{Strong}}(3) \otimes \text{SU}_{\text{weak}}(2) \otimes \text{U}_Y(1)$ gauge theory without Higgs mechanism. In addition, without chiral-density-wave order and chiral-spacetime-crystal, there doesn't exist chiral-gravity at all.

H. Discussion and conclusion

In this section, we propose a new quantum gravity theory – chiral-gravity theory, of which quantum fields and curved spacetime are unified into a single phenomenon – composite knot-crystal. The most striking picture is that the origin of curved spacetime and gravitation interaction and the origin of chirality for neutrinos and weak interaction are just the different sides of the same coin: chiral-spacetime-crystal from $\delta \neq 0$. Let us summarize the key points of chiral-gravity theory:

1. **Chiral-spacetime-crystal from a very-slowly varying chiral-density-wave order:** To generate the gravitational interaction between knots, we may just consider a very-slowly varying chiral-density-wave order on a double-helix knot-crystal. The chiral-density-wave order can be regarded as a composite knot-crystal for chiral-spacetime-crystal,

$$\begin{aligned} &\text{Non-uniform chiral-density-wave order} \\ &\text{on double-helix knot-crystal} \\ &\implies \text{deformed chiral-spacetime-crystal} \\ &\implies \text{curved spacetime.} \end{aligned} \quad (660)$$

On the contrary, without a very slowly varying chiral-density-wave order, there doesn't exist gravitational interaction and curved spacetime anymore.

2. **Double-role of chiral angles:** For composite knots of the chiral-spacetime-crystal, on the one hand, chiral angles $(\vec{\Theta}, \Theta_t)$ denote the internal degrees of freedom of composite knots; on the other hand, chiral angles $(\vec{\Theta}, \Theta_t)$ play the role of scales of a quantized spacetime with diffeomorphism invariance, as

$$\text{Chiral angles} \iff (\vec{\Theta}, \Theta_t) \implies \text{Spacetime coordinates.} \quad (661)$$

The deformation of chiral-spacetime-crystal leads to the changing of "motions" for all knots on it. For this reason, the chiral-spacetime-crystal becomes an emergent curved spacetime.

3. **Topological chiral-orbital coupling effect:** In chiral-gravity theory, a topological interplay effect between chiral order and orbital motion is called topological chiral-orbital coupling effect, i.e.,

$$\begin{aligned} &\text{Matter (knots)} \iff \text{Curved spacetime} \\ &\quad (\text{chiral-spacetime-crystal}). \end{aligned} \quad (662)$$

On the one hand, the deformation of the chiral-density-wave-order (or chiral-spacetime-crystal) leads to the changes of knot-motions that can be denoted by curved spacetime; on the other hand, the linear knots trapping "magnetic" charges and topological lattice-defects deform the chiral-density-wave-order (chiral-spacetime-crystal) that indicates matter may curve spacetime. In mathematic, the Einstein-Hilbert action S_{EH} becomes a topological mutual BF term S_{MBF} in chiral-gravity theory.

E. Witten had made a strong claim about emergent gravity, "*whatever we do, we are not going to start with a conventional theory of non-gravitational fields in Minkowski spacetime and generate Einstein gravity as an emergent phenomenon.*" He pointed out that gravity could be emergent only if the notion on the spacetime on which diffeomorphism invariance is simultaneously emergent. In knot physics, chiral-gravity is an emergent Einstein gravity, in

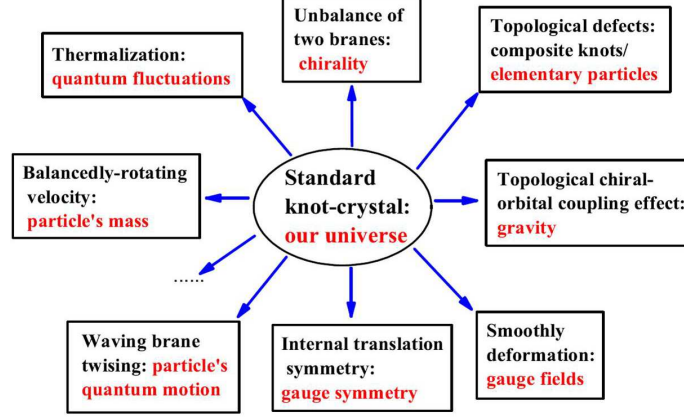


FIG. 24: An illustration of knot physics : physics of standard knot-crystal under infinite thermalization condition

which diffeomorphism invariance and Lorentz invariance are indeed simultaneously emergent. The low energy effective theory on standard knot-crystal exactly reproduces the physics of the general relativity. Furthermore, chiral-gravity from knot physics is naturally free from all troubles of divergences or singularities in conventional gravity theory. The renormalizability is guaranteed from the very beginning. In addition, for chiral-gravity theory from knot physics, there don't exist any types of gravitational singularities.

V. CONCLUSION

In this paper, a new TOE theory – *knot physics* is developed to understand our universe. See the illustration in Fig.24. In knot physics, our universe is a knot-crystal, a periodic entanglement-pattern between two 3-branes (three dimensional branes/manifolds). The collective motions of knot-crystal are described by fermionic elementary particles and gauge fields. Fermionic elementary particles are topological excitations that correspond to different types of knots. All fundamental interactions are unified into single simple framework – the standard knot-crystal of which the low energy effective theory not only reproduces the Standard model – an $SU_{\text{Strong}}(3) \otimes SU_{\text{weak}}(2) \otimes U_Y(1)$ gauge theory but also leads to the physics of general relativity. In addition, this TOE yields a deep understanding of the all kinds of elementary particles within different entanglement-pattern between two 3-branes.

By using knot physics, we answer the six questions. 1) *what is matter?* Answer: matter is knot that is half winding between two 3-branes in five dimensional space; 2) *How matter moves?* Answer: quantum motion of matter comes from waving brane-twisting in extra dimensions; 3) *what is quantum field?* Answer: quantum field is a (composite) knot-crystal; 4) *what is (local) gauge symmetry?* Answer: (local) gauge symmetry is the generalized (internal) translation symmetry of composite knot-crystals; 5) *what is gravity?* Answer: gravity is topological chiral-orbital coupling effect between knots and 3D double-helix knot-crystal; 6) *what is curved spacetime?* Answer: curved spacetime is deformed chiral-spacetime-crystal (a 3D double-helix knot-crystal with very-slowly varying chiral-density-wave order).

For TOE of string theory or that from emergent physics in a complicated many-body system, quantum mechanics is assumed to be unchanging. However, in knot physics, quantum mechanics is replaced by a new mechanics – knot mechanics. In knot physics almost all aspects of our universe are unified into single simple phenomenon.

In the future, we will develop a complete knot physics and point out several important research directions. The first research direction is cosmology. We will study the black hole and origin of our universe. Along this direction, dark energy and dark matter are also important issues; the second one is particle physics. We will calculate the value of the elements of Cabibbo-Kobayashi-Maskawa mass matrix[36] and learn the mechanism of weak charge-parity (CP) violence. Along this research direction, quark confinement, including the energy scale of string tension and the spontaneous chiral symmetry breaking in quantum chromodynamics is another important issue; the third one is hadron physics. The most important issue is the detailed structures of proton and neutron; the fourth research direction is strongly correlated physics. Knot physics may provides new approach to understand the quantum orders in complex many-body physics[3]. Except for above research fields, it is interesting to study the relationship between knot physics and other TOEs (for example, superstring theory, quantum loop theory, twistor theory) that may be

alternative formalisms of knot physics.

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